An introduction to modal logic

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2/5/2017

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- Alethic Reading: $\Box\phi$ means 'ϕ is necessary' and $\Diamond\phi$ means 'ϕ is possible'.
- Deontic Reading: $\Box\phi$ means 'ϕ is obligatory' and $\Diamond\phi$ means 'ϕ is permitted'. In this literature, typically 'O' is used instead of '$\Box$' and 'P' instead of '$\Diamond$'.
- Epistemic Reading: $\Box\phi$ means 'ϕ is known' and $\Diamond\phi$ means 'ϕ is consistent with the current information'. In this literature, typically 'K' is used instead of '$\Box$' and 'L' instead of '$\Diamond$'.
- Temporal Reading: $\Box\phi$ means 'ϕ will always be true' and $\Diamond\phi$ means 'ϕ will be true at some point in the future'

Definition

(The Basic Modal Language) Let $\mathbb{P} = \{\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, ...\}$ be a set of sentence letters, or atomic propositions. We also include two special propositions $\top$ and $\bot$ meaning 'true' and 'false' respectively. The set of well-formed formulas of modal logic is the smallest set generated by the following grammar: $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, ... \mid \top \mid \bot \mid \neg A \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A \mid \Diamond A$

Examples

Modal formulas include: $\Box\bot, \mathbb{P}_0 \rightarrow \Diamond(\mathbb{P}_1 \wedge \mathbb{P}_2)$.

Models

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Truth conditions:

1. $\models_\alpha^\mathcal{M} \mathbb{P}_i$ iff $\alpha \in P_i$
2. $\models_\alpha^\mathcal{M} \top$
3. $\not\models_\alpha^\mathcal{M} \bot$
4. $\models_\alpha^\mathcal{M} \neg A$ iff not $\models_\alpha^\mathcal{M} A$
5. $\models_\alpha^\mathcal{M} A \vee B$ iff either $\models_\alpha^\mathcal{M} A$ or, $\models_\alpha^\mathcal{M} B$,or both
6. $\models_\alpha^\mathcal{M} A \wedge B$ iff both $\models_\alpha^\mathcal{M} A$, and $\models_\alpha^\mathcal{M} B$
7. $\models_\alpha^\mathcal{M} A \rightarrow B$ if $\models_\alpha^\mathcal{M} A$, then $\models_\alpha^\mathcal{M} B$
8. $\models_\alpha^\mathcal{M} \Box A$ iff for every β in $\mathcal{M}, \models_\beta^\mathcal{M} \mathcal{A}$
9. $\models_\alpha^\mathcal{M} \Diamond A$ iff for some β in $\mathcal{M}, \models_\beta^\mathcal{M} \mathcal{A}$

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Everywhere holds:

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Propositional logic

Relationship to propositional logic:

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Relationship to propositional logic: modal logic includes propositional logic.

1. If A is a tautology, then $\models A$
2. Propositional correct patterns are still applied in modal logic

$$(MP)\ \text{If } \models A \rightarrow B \text{ and } \models A, \text{ then } \models B$$

Invalid sentences

Some invalid sentences:

Example

$A \rightarrow \Box A$

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$A \to \Box A$

countermodel: $W = \{\alpha, \beta\}$, $P_n = \{\alpha\}, n = 0, 1, 2, \ldots$

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Example

$\Box(A \vee B) \to (\Box A \vee \Box B)$

Axiomatization

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Axioms:

1. T. $\Box A \to A$
2. 5. $\Diamond A \to \Box \Diamond A$
3. K. $\Box(A \to B) \to (\Box A \to \Box B)$
4. $Df\Diamond$. $\Diamond A \leftrightarrow \neg\Box\neg A$
5. PL. A, where A is a tautology.

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$\vdash A$ means sentence A is a theorem

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$$RPL. \quad \frac{A_1, A_2, ..., A_n}{A}, n \geq 0$$

where the inference from $A_1, ..., A_n$ to A is propositionally correct

New theorems

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$T\Diamond.\ \ A \rightarrow \Diamond A$

Proof:

$$
\begin{array}{lll}
1. & \Box\neg A \rightarrow \neg A & \text{T} \\
2. & A \rightarrow \neg\Box\neg A & 1, \text{PL} \\
3. & \Diamond A \leftrightarrow \neg\Box\neg A & Df\Diamond \\
4. & A \rightarrow \Diamond A & 2, 3, \text{PL}
\end{array}
$$

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Since $\Box A \to A$ and $A \to \Diamond A$ are theorems, by PL so is $\Box A \to \Diamond A$.

B. $A \to \Box\Diamond A$
Proof:

$$\begin{array}{lll} 1. & \Diamond A \to \Box\Diamond A & 5 \\ 2. & A \to \Diamond A & T\Diamond \\ 3. & A \to \Box\Diamond A & 1, 2, PL \end{array}$$

Two more rules of inference:

$$RM. \quad \frac{A \rightarrow B}{\Box A \rightarrow \Box B}$$

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$$RE. \quad \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

Proof for RM:

1.	$A \to B$	hypothesis
2.	$\Box(A \to B)$	$1, \mathrm{RN}$
3.	$\Box(A \to B) \to (\Box \to \Box B)$	K
4.	$\Box A \to \Box B$	$2, 3\mathrm{PL}$

Another theorem:

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1.	$\Diamond\neg A \leftrightarrow \neg\Box\neg\neg A$	$Df\Diamond$
2.	$\Box\neg\neg A \leftrightarrow \neg\Diamond\neg A$	1, PL
3.	$A \leftrightarrow \neg\neg A$	PL
4.	$\Box A \leftrightarrow \Box\neg\neg A$	3, RE
5.	$\Box A \leftrightarrow \neg\Diamond\neg A$	2, 4, PL

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$$4.\quad \Box A \to \Box\Box A$$

1.	$\Diamond\Box A \to \Box A$	$5\Diamond$
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4.	$\Box A \to \Box\Box A$	2, 3, PL

with dual

$$4\Diamond.\ \ \Diamond\Diamond A \to \Diamond A$$

Another rule of inference:

$$RK. \quad \frac{(A_1 \wedge A_2 ... \wedge A_n) \to A}{(\Box A_1 \wedge \Box A_2 ... \wedge \Box A_n) \to \Box A}, n \geq 0$$

Soundness and completeness

All theorems are valid and the rules of inference preserve validity.

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Not the only way to axiomatize S5: one of the best known, is RN together with T, B, 4, K, Df$\Diamond$ as axioms.

Other systems

Some of the most popular systems are:
K := K + N
K4 := K + 4
T := K + T
S4 := T + 4
S5 := S4 + 5
D := K + D.

Frames

Definition

(Frame) A pair $< W, R >$ with W a nonempty set of states (worlds) and $R \subseteq W \times W$ is called a *frame*. Given a frame $F =< W, R >$, we say the (Kripke) model $\mathcal{M}$ *is based on the frame* $F =< W, R >$ if $\mathcal{M} =< \mathcal{W}, \mathcal{R}, \mathcal{V} >$ for some valuation V.

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An accessibility relation $R \subseteq G \times G$ is:

- *reflexive* iff wRw, for every $w \in G$
- *symmetric* iff wRu implies uRw, for all $w, u \in G$
- *transitive* iff wRu and uRv together imply wRv, for all $w, u, v \in G$.
- *serial* iff, for each $w \in G$ there is some $u \in G$ such that wRu.
- *Euclidean* iff, for every $u, t \in G$, and $w \in G$, wRu and wRt implies uRt (note that it also implies: tRu)

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If the accessibility relation R is reflexive and Euclidean, R is provably symmetric and transitive as well.
Hence for models of S5, R is an equivalence relation, because R is reflexive, symmetric and transitive.