Opinion Dynamics

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Overview

Motivation

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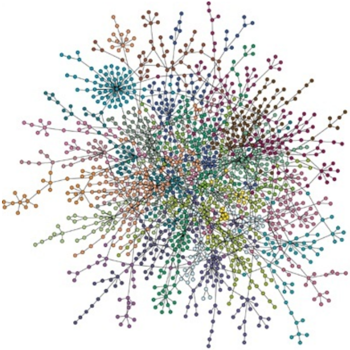

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Opinion Formation

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- For an agent i, G shows us which agents influence i and by how much.
- The weight $w_{ij} \geq 0$ of an edge (i, j) represents j's influence over i.

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- If $w_{ii} > 0$, this implies $i \in \mathcal{N}_i$.

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2. Opinion Formation Models
   - Linear Models
   - Non-linear (Coevolutionary) Models
3. Conclusion

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Two important variations:

- The **undirected** DeGroot model. $\forall i, j \ w_{ij} = w_{ji} \implies \boldsymbol{A} = \boldsymbol{A}^T$.

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- The **directed** DeGroot model. No assumption on w_{ij}, w_{ji}.

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- This implies that $\exists t_0 : \boldsymbol{A}^{t_0}$ has only positive elements.
- Since $\boldsymbol{x}^* = \boldsymbol{A}\boldsymbol{x}^*$, we call $\boldsymbol{x}^*$ the **Nash equilibrium** of the model.

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- Since $\boldsymbol{A}$ is stochastic, $\rho(\boldsymbol{A}) = \lambda_1 = 1$ is a unique eigenvalue and it corresponds to $\boldsymbol{q}_1 = \frac{1}{n}\mathbf{1} = [\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}]$.

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- The eigenvectors $\boldsymbol{q}_i$ of $\boldsymbol{A}$ are orthonormal and linearly independent, thus they span $\mathbb{R}^n$.
- We can write $\boldsymbol{x}^T(0) = \sum_{i=1}^{n} c_i \boldsymbol{q}_i$, for some $c_i \in \mathbb{R}$.

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- Therefore, the Nash equilibrium is a state in which all agents share the same opinion.
- We call such an equilibrium a **consensus**.
- Because $|\lambda_2| \geq |\lambda_i| \ \forall 3 \leq i \leq n$, the convergence rate to $\boldsymbol{x}^*$ is exponential in the order of λ_2.

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- $\boldsymbol{s}$ is the vector of the agents' intrinsic opinions.
- Assuming $\boldsymbol{x}(0) = \boldsymbol{s}$, we get:

$$\boldsymbol{x}(t) = \boldsymbol{A}^t \boldsymbol{s} + \sum_{k=0}^{t-1} \boldsymbol{A}^k \boldsymbol{B} \boldsymbol{s}$$

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- In 2012, Ghaderi and Srikant proved that the undirected FJ model converges to $\boldsymbol{x}^*$ in $\mathcal{O}\left(\frac{ln(n/\gamma)}{1-\rho(\boldsymbol{A})}\right)$ time.

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- If $w_{ii} = 0$, we call agent i a **non-stubborn** agent.
- $\boldsymbol{x}^*$ is a convex combination of the initial opinions of stubborn agents.

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- The opinions of the agents and the underlying social network coevolve, with one affecting the other.
- Much more interesting models, but linear techniques cannot be applied, making them considerably more difficult to analyze.

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$$\boldsymbol{x}(t) = \boldsymbol{A}(t, \boldsymbol{x}(t))\boldsymbol{x}(t-1)$$

- For convenience, $\boldsymbol{A}_t := \boldsymbol{A}(t, \boldsymbol{x}(t))$.
- The (i, j) element of each $\boldsymbol{A}_t$ is:
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The Hegselmann - Krause Model

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The Hegselmann - Krause Model

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If for two agents i and j we have $|x_i(t) - x_j(t)| > \varepsilon$ at time t, we call this event a **split** at time t, because it leads to $i \notin \mathcal{N}_j(t+1, \varepsilon)$ and $j \notin \mathcal{N}_i(t+1, \varepsilon)$.

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3. If a specific instance of the HK model reaches consensus, this implies that $|x_i(t) - x_j(t)| \leq \varepsilon$ for all agents i, j and all times $t \geq 0$.

The Hegselmann - Krause Model - Results

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- They also provided a lower bound of $\Omega(n^2)$ for the 1-dimensional case.

The Deffuant - Weisbuch Model

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- While highly unlikely, it is possible for an agent k to not be chosen for all times t up to a fixed time t_0, therefore having $x_k(t) = x_k(0) \ \forall t \leq t_0$.
- In this case, agent k behaves as a fully-stubborn agent that never updates his opinion.

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- In 2012, Chazelle provided an exponential upper bound on the convergence time.
- However, no significant upper or lower bounds are known on the convergence time of the DW model.

Overview

1. Introduction
2. Opinion Formation Models
   - Linear Models
   - Non-linear (Coevolutionary) Models
3. Conclusion

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- Considerable work is being done on several variations of the HK model.
- Chazelle, Munagala, Fotakis et al have provided significant results on the convergence properties of such variations.

QUESTIONS ?