Introduction to Lattices

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Contents

1 Lattices in $\mathbb{R}^m$

2 The LLL Algorithm

3 Babai's Nearest Plane Algorithm

4 Complexity Results

Why lattices?

- Algebraic Algorithms

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- Algebraic Algorithms
- Combinatorial Optimization
- Complexity
- Cryptanalysis
- Cryptography
  - Very efficient
  - Worst case security guarantees
  - No known quantum attacks
  - Exotic constructions

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Definition

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Given n linearly independent vectors $b_1, \ldots, b_n \in \mathbb{R}^m$ the lattice generated by them is the set

$$\mathcal{L}(b_1, \ldots, b_n) = \{\sum_{i=1}^{n} x_i b_i \mid x_i \in \mathbb{Z}\}$$

Denoting $B = [b_1 \; b_2 \; \ldots \; b_n]$ equivalently we have

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- B is the basis of the lattice.
- m is its dimension.
- n is its rank.
- A lattice is full rank if $m = n$.
- The span of the lattice is the linear span of its basis.

- A lattice has many (infinite) equivalent bases.
- We next define the fundamental parallilepiped with respect to a basis B.

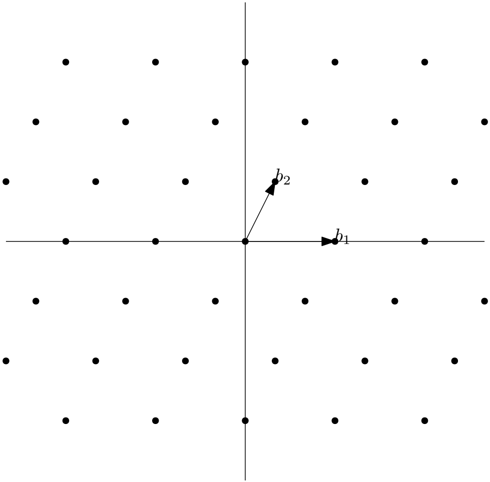
Definition

Fundamental Parallilepiped For a basis B the fundamental parallilepiped is the set

$$\mathcal{P}(B) = \{Bx \mid x \in [0, 1)\}$$

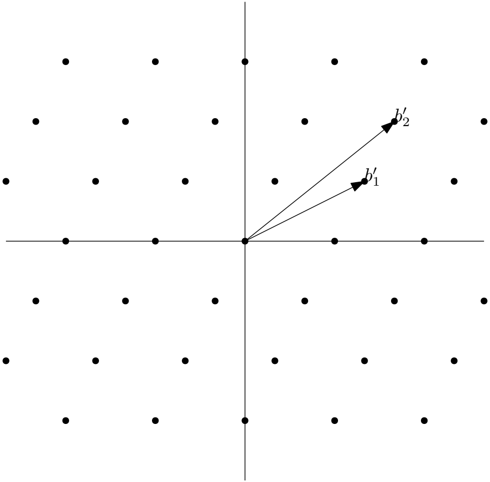
- Placing a copy of $\mathcal{P}(B)$ in every lattice point we partition $\text{span}(B)$.

A Lattice in $\mathbb{R}^2$



A lattice $\Lambda = \mathcal{L}(b_1, b_2) \subseteq \mathbb{R}^2$.

A different basis for the same lattice



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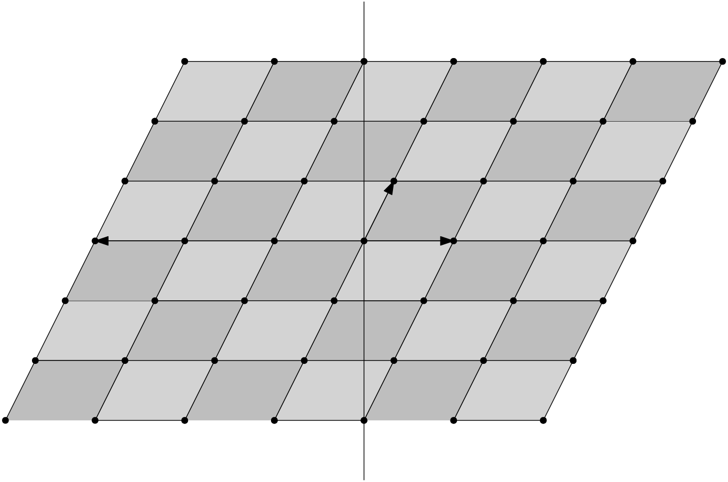
Fundamental Parallilepiped I



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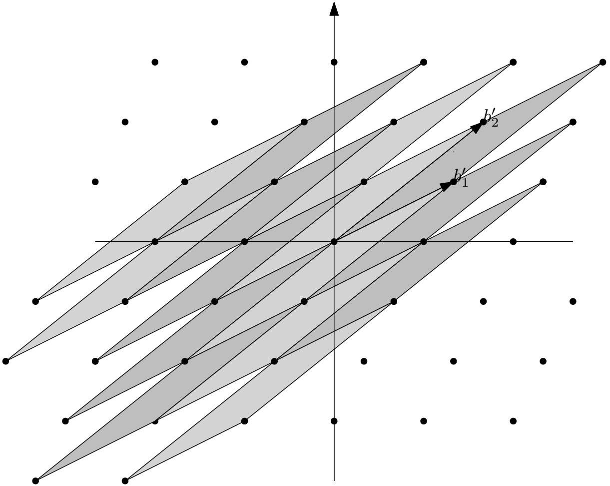

Characterizing a Basis

Lemma

Let Λ be a lattice and $b_1, \ldots, b_n \in \Lambda$ be n linearly independent vectors. Then $\Lambda = \mathcal{L}(b_1, \ldots, b_n)$ iff $\mathcal{P}(b_1, \ldots, b_n) \cap \Lambda = \{0\}$.

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Proof.

($\Rightarrow$) Suppose $x \in \mathcal{P}(b_1, \ldots, b_n) \cap \Lambda = \{0\}$. Then

$$x = Bz \text{ for } z \in \mathbb{Z}^n$$

$$x = By \text{ for } y \in [0, 1)^n$$

Since $b_1, \ldots, b_n$ are linearly independent we get $z = 0$.

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Since $b_1, \ldots, b_n$ are linearly independent we get $z = 0$.
($\Leftarrow$) Suppose $x \in \Lambda$. Then $x = By$ for $y \in \mathbb{R}^n$. Now $x' = B(y - \lfloor y \rfloor) \in \Lambda$. We get that $y = \lfloor y \rfloor$.

■

Equivalent Basis

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Suppose $B, D \in \mathbb{R}^{m \times n}$ rank n matrices. Then $\mathcal{L}(B) = \mathcal{L}(D)$ iff $D = BU$ for $U \in \mathbb{Z}^{n \times n}$ unimodular matrix.

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($\Rightarrow$) If $\mathcal{L}(B) = \mathcal{L}(D)$ then each column $b_i \in \mathcal{L}(D)$ so $b_i = Du_i$. In matrix form we have $B = DU$. Similarly $D = BV$. Then we have $B^TB = U^TV^TB^TBVU$ so $\det(B^TB) = \det(U^TV^T)\det(B^TB)\det(VU)$ and so $\det(VU)^2 = 1$ and we get $\det(V)\det(U) = \pm 1$.

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($\Leftarrow$) Suppose $D = BU$. Then $D \subseteq \mathcal{L}(B)$. Also $B = DU^{-1}$ so $B \subseteq \mathcal{L}(D)$. We get $\mathcal{L}(B) = \mathcal{L}(D)$. ∎

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The determinant of a lattice Λ is the n-dimentional volume of a fundamental parralilepiped $\mathcal{P}(B)$, that is $\det(\Lambda) = \sqrt{\det(B^T B)}$.

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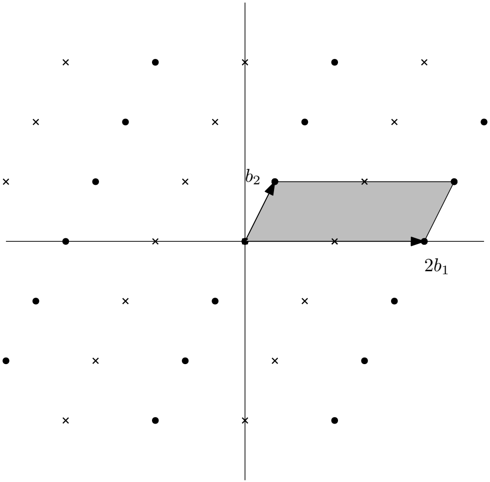
- It expresses the *density* of a lattice.

A Sublattice



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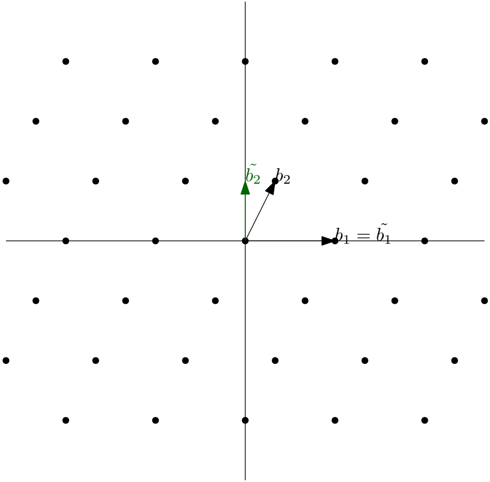
- The order of the input matters.

An example of GSO



Gram Schmidt Orthogonormal Basis

Let $b_1, \ldots, b_n \in \mathbb{R}^m$. If we normalize the GSO vectors we get an orthonormal basis

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In this basis we have

$$B = \begin{bmatrix} ||\tilde{b_1}|| & \mu_{2,1}||\tilde{b_1}|| & \ldots & \mu_{n,1}||\tilde{b_1}|| \\ 0 & ||\tilde{b_2}|| & \ldots & \mu_{n,2}||\tilde{b_2}|| \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & ||\tilde{b_n}|| \\ 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix}$$

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From this we can easily get that $\det(\mathcal{L}(B)) = \prod_{i=1}^{n} ||\tilde{b_i}||$

Successive Minima

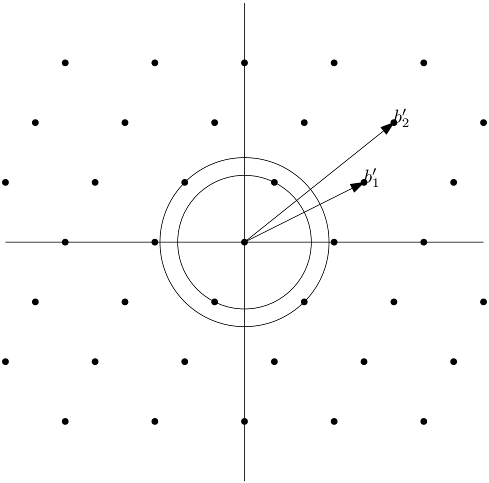
We define the i-th successive minima of Λ as the radius of the smallest ball that contains that contains i linearly independent lattice points. More formally

Definition

The i-th successive minima of Λ is

$$\lambda_i(\Lambda) = \inf\{r \mid \dim(\mathrm{span}(\Lambda \cap \overline{\mathbf{B}}(0, r))) > i\}$$

Successive Minima Example



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- Lattices are discrete structures.

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For any full rank lattice Λ *and any measurable* $S \subseteq \mathbb{R}^n$ *with* $\mathrm{vol}(S) > \det(\Lambda)$, *there exist* $z_1, z_2 \in S$ *such that* $z_1 - z_2 \in \Lambda$.

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- $(z + x) - (z + y) = x - y \in \Lambda$.

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For any full rank lattice Λ *and any centrally symmetric, convex set* $S \subseteq \mathbb{R}^n$ *if* $\mathrm{vol}(S) > 2^n \det(\Lambda)$, *then* S *contains a non zero lattice point.*

Proof.

- Define $S' = \frac{1}{2}S = \{x \mid 2x \in S\}$.

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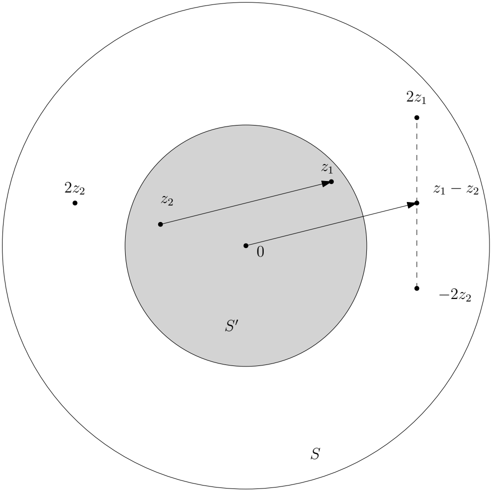
- We have $2z_1, -2z_2 \in S$ and we get $z_1 - z_2 \in S$

■

Minkowski's Convex Body Theorem



Bounding Successive Minima

By selecting appropriate sets (ball and ellipsoid respectively) we can deduce the following upper bounds for the succesive minima using the previous theorem.

$$\lambda_1(\Lambda) \leq \sqrt{n} \cdot \det(\Lambda)^{\frac{1}{n}}$$

$$(\prod_{i=1}^{n} \lambda_i(\Lambda))^{\frac{1}{n}} \leq \sqrt{n} \cdot \det(\Lambda)^{\frac{1}{n}}$$

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Things get harder when geometry comes to play.

Shortest Vector Problem

- **SearchSVP$_\gamma$**: Given $B \in \mathbb{Z}^{m \times n}$ find $v \in \mathcal{L}(B)$ such that $v \neq 0$ and $\|v\| \leq \gamma \cdot \lambda_1(\mathcal{L}(B))$.

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For the general case it is an open problem if this holds.

Closest Vector Problem

- **SearchCVP**$_\gamma$: Given $B \in \mathbb{Z}^{m \times n}$ and $t \in \mathbb{Z}^m$ find $v \in \mathcal{L}(B)$ such that $\|v - t\| \leq \gamma \cdot \text{dist}(t, \mathcal{L}(B))$.

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- It is also used for approximating **CVP**.
- Used for many problems, namely many algebraic problems, combinatorial optimization, cryptanalisis.

$\delta-$LLL Reduced Basis

Definition

A basis $B = [b_1 \ b_2 \ \ldots \ b_n]$ is a $\delta-$LLL reduced basis if

1. forall $i \in [n]$, $j < i$ it holds that $|\mu_{i,j}| \leq \frac{1}{2}$
2. forall $i \in [n]$ it holds that $\delta \|\tilde{b}_i\|^2 \leq \|\mu_{i+1,i}\tilde{b}_i + \tilde{b}_{i+1}\|^2$.

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$$\|\tilde{b}_{i+1}\|^2 \geq (\delta - \mu_{i+1,i}^2)\|\tilde{b}_i\|^2 \geq (\delta - \frac{1}{4})\|\tilde{b}_i\|$$

That is $\tilde{b}_{i+1}$ is not much shorter than $\tilde{b}_i$

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- LLL produces a $\delta-$LLL reduced basis.

$\delta-$LLL Reduced Basis

Consider the orthonormal basis produced by GSO. A $\delta-$LLL reduced basis looks like this

$$\begin{bmatrix} \|\tilde{b}_1 & \leq \frac{1}{2}\|\tilde{b}_1\| & \cdots & \leq \frac{1}{2}\|\tilde{b}_1\| \\ 0 & \|\tilde{b}_2\| & \cdots & \leq \frac{1}{2}\|\tilde{b}_2\| \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \leq \frac{1}{2}\|\tilde{b}_{n-1}\| \\ 0 & 0 & \dots & \|\tilde{b}_n\| \end{bmatrix}$$

Approximating **SVP** with LLL

Theorem

Suppose $b_1, \ldots, b_n$ is a $\delta-$LLL reduced basis. Then

$$\|b_1\| \leq \left(\frac{2}{\sqrt{4\delta - 1}}\right)^{n-1} \lambda_1(\mathcal{L}(B))$$

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After a few calculations we get that forall i

$$\|\tilde{b}_1\| \leq \left(\delta - \frac{1}{4}\right)^{\frac{n-1}{2}}\|\tilde{b}_i\|$$

and since $\lambda_1(\mathcal{L}(B)) \geq \min_i \|\tilde{b}_i\|$ we get the result. ∎

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For $\delta = \frac{3}{4}$ this gives a $2^{\frac{n-1}{2}}$ approximation ratio.

The LLL Algorithm

1. **Start** Compute $\tilde{b}_1, \ldots, \tilde{b}_n$
2. **Reduction**
   for $i = 2$ to n
   for $j = i - 1$ to 1
   $c_{i,j} = \lfloor \frac{\langle b_i, \tilde{b}_j \rangle}{\langle \tilde{b}_j, \tilde{b}_j \rangle} \rceil$
   $b_i \leftarrow b_i - c_{i,j} b_j$
3. **Swap**
   if there exists i s.t $\delta \|\tilde{b}_i\|^2 > \|\mu_{i+1,i} \tilde{b}_i + \tilde{b}_{i+1}\|^2$
   $b_i \leftrightarrow b_{i+1}$
   goto start

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- We need to show that the number of iterations is polynomial.

Bounding iterations of LLL

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- Initially the size of $\mathcal{D}_B$ is polynomial. This is because

$$\mathcal{D}_B = \prod_{i=1}^{n} \|\tilde{b_1}\| \cdots \|\tilde{b_i}\| = \|b_1\|^n \|b_2\|^{n-1} \cdots \|b_n\| \leq \max_i \|b_i\|^{\frac{n(n+1)}{2}}$$

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- We will show that $\mathcal{D}_B$ decreases by a constant factor in each iteration.

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$$\frac{\mathcal{D}'_{B,i}}{\mathcal{D}_{B,i}} = \frac{\det(\mathcal{L}(b_1, \ldots, b_{i-1}, b_{i+1}))}{\det(\mathcal{L}(b_1, \ldots, b_i))}$$

$$= \frac{(\prod_{j=1}^{i-1} \|\tilde{b}_j\|)\|\mu_{i+1,i}\tilde{b}_i + \tilde{b}_{i+1}\|}{\prod_{j=1}^{i} \|\tilde{b}_j\|} = \frac{\|\tilde{b}_j\| \cdot \|\mu_{i+1,i}\tilde{b}_i + \tilde{b}_{i+1}\|}{\|\tilde{b}_i\|} \leq \sqrt{\delta}$$

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- So we can have at most $\log_{\frac{1}{\sqrt{\delta}}} \mathcal{D}_B$ iteration which is polynomial.

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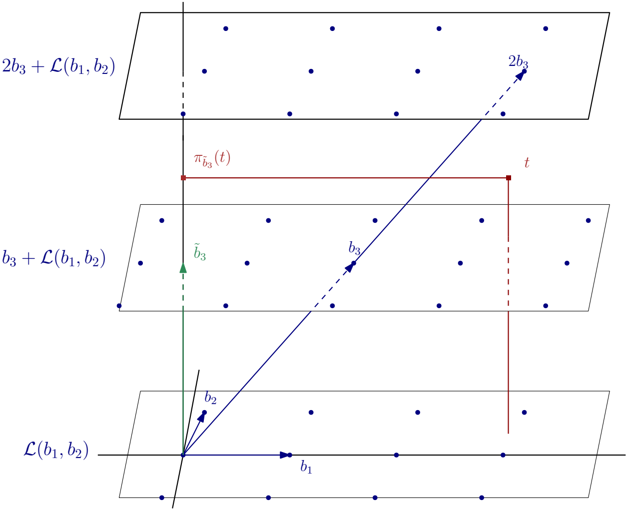
- We will present the algorithm and omit its analysis.

The Nearest Plane Algorithm

- Compute a $\delta-$LLL reduced basis.
- $b \leftarrow t$
  for $j = n$ to 1
  $c_j = \lfloor \frac{\langle b, \tilde{b}_j \rangle}{\langle \tilde{b}_j, \tilde{b}_j \rangle} \rceil$
  $b \leftarrow b - c_j b_j$
- **return** $t - b$

Geometric Illustration of the Algorithm



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- If $\langle \{a_1, \ldots, a_n\}, S\rangle \in$ **SubsetSum** then suppose that $\sum_{i \in A} a_i = S$. Now take $y = Bz$ where z is vector with its j-th coordinate equal to 1 if $j \in A$ and 0 otherwise. Then $\|Bz - t\| = \|[0 \ \pm 1 \ \ldots \ \pm 1]^T\| = \sqrt{n}$ and so $\langle B, t, r\rangle \in \textbf{PromiseCVP}_1$.

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- If $\langle B, t, r \rangle \in \mathbf{PromiseCVP}_1$. Assume $x \in \mathcal{L}(B)$ such that $\|x - t\| \leq \sqrt{n}$ The last n coordinates are even, so subtracting 1 gives at least $\sqrt{n}$. It must be the case that the first coordinate is S.

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- Note that if we find the closest vector to $t + v$ for $v \in \mathcal{L}$ we are done.

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We have that $\mathcal{L}(B) = \mathcal{L}(B') \cup \mathcal{L}(B' + b_1)$. So the distance to from t to the new subblatice or the shifted sublattice is r.

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We dont know how to generalize this for the gap versions of the problems.

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- So in the reduction, we delete a set of lattice points.

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- If any returns YES we return YES otherwise we return NO.

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- if $(B, r) \notin$ **PromiseSVP**$_\gamma$ then $\lambda_1(\mathcal{L}(B)) > \gamma \cdot r$. So any lattice vector v has length $\|v\| > \gamma \cdot r$. Now suppose that for some i the oracle returned YES. Then there exists $v \in \mathcal{L}(B_i)$ s.t. $\|v - b_i\| \leq r$. But this is a lattice point in $\mathcal{L}(B)$ which is a contradiction.

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- if $(B, r) \in \mathbf{PromiseSVP}_\gamma$ then $\lambda_1(\mathcal{L}(B)) \leq r$. Let v be the smallest vector. Then $v = a_1 b_1 + \ldots + a_n b_n$ for some a_i odd. Then $b_i + v \in \mathcal{L}(B_i)$ and its distance from b_i is less than r so the oracle must return YES.

The end!

Thank you! Questions?