

Approximation Algorithms

Metric Steiner Tree

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Steiner Tree problem

- Given an undirected graph $G=(V,E)$ with non negative edge costs and whose vertices are partitioned into two sets, required (denoted by R) and Steiner (denoted by S)
- Find a minimum cost tree in G containing all vertices in R and any subset of vertices in S

metric Steiner Tree problem

- Additional restriction: the graph is complete and the edge costs satisfy the triangle inequality, i.e. for any three vertices u,v,w in V

$$\text{cost}(u,v) \leq \text{cost}(u,w) + \text{cost}(v,w)$$

Approximation Factor Preserving Reduction from Steiner Tree problem to metric Steiner Tree problem

- **Theorem:** Any $a(n)$ -approximation algorithm for metric Steiner Tree can be transformed to an $a(n)$ -approximation algorithm for Steiner Tree
- **Proof:** We will transform an instance I of the Steiner Tree problem to an instance I' of the metric Steiner Tree problem.

We construct a complete undirected graph G' on vertex set V . We keep R' and S' as in I . The cost between vertices u', v' in G' is equal to the cost of the shortest $u-v$ path in G (G' is the metric closure of G).

$\text{OPT}(I') \leq \text{OPT}(I)$, since $\text{cost}(u', v') \leq \text{cost}(u, v)$

Given a Steiner Tree T' in I' , we will construct a Steiner Tree in I by replacing each edge of T' by its corresponding path in G and deleting the edges that create cycles.

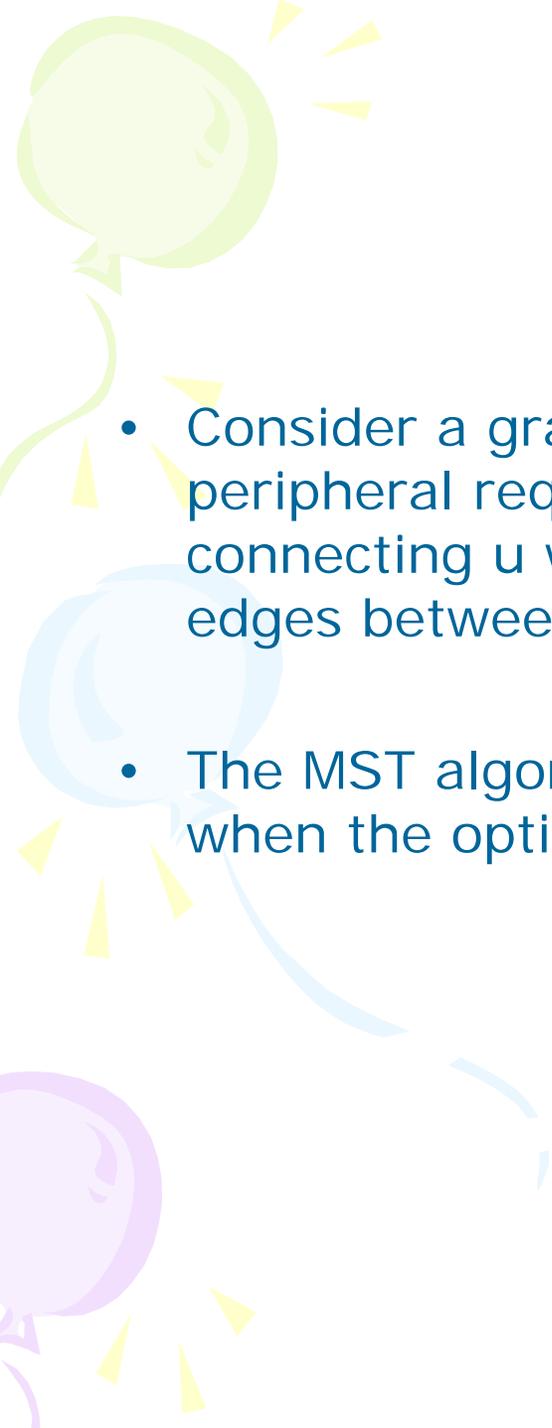
- all the required vertices are connected

$$\text{SOL}(I) \leq \text{SOL}(I') \leq a(n)\text{OPT}(I') \leq a(n)\text{OPT}(I)$$

2-approximation algorithm for metric Steiner Tree problem

- **Note:** Any minimum spanning tree (MST) on R is a feasible solution for metric Steiner Tree problem
- **Algorithm:** Find a MST on R .
- **Theorem:** $\text{cost}(\text{MST on } R) \leq 2 \text{ OPT}$.
- **Proof:** Technique similar to TSP. Consider a Steiner tree of cost OPT . Double the edges to obtain an Euler graph (its cost is 2 OPT). Find an Euler graph and obtain a Hamilton cycle by “shortcutting” Steiner vertices and visited vertices. Delete one edge of the HC to obtain a tree that spans R .

The shortcuts do not increase the cost of the solution produced. Therefore, this solution has cost at most 2 OPT .



Tight example

- Consider a graph with one central Steiner vertex u and n peripheral required vertices. The cost of the edges connecting u with the n nodes is 1 and the cost of the edges between the n nodes is 2.
- The MST algorithm produces a solution with cost $2(n-1)$, when the optimal cost is n .