Arithmetical Hierarchy

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$\mu\Pi\lambda\forall$

Algorithms and Complexity II

Arithmetical Hierarchy

Definitions

For $k \geq 0$,

- Σ_k^0 is the class of languages

  $$L = \{x | \exists x_1 \forall x_2 \ldots Q_k x_k R(x_1, \ldots, x_k, x)\}$$

  where R is recursive relation, and

  $$Q_k = \begin{cases} \exists, & \text{if } k \text{ is odd} \\ \forall, & \text{if } k \text{ is even} \end{cases}$$

  and also $x_i, \forall i \in \{1, \ldots, k\}$ are tuples of natural numbers.
- $\Pi_k^0 = \mathbf{co}\Sigma_k^0$
- $\Delta_k^0 = \Sigma_k^0 \cap \Pi_k^0$

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- $\Pi_k^0 = \mathbf{co}\Sigma_k^0$
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More definitions...

- $L \in \Sigma_k^0 \Rightarrow \bar{L} = \{x|\neg(\exists x_1 \forall x_2 \dots Q_k x_k R(x_1, x_2, \dots, x_k, x))\} \Rightarrow \bar{L}\{x|\forall x_1 \exists x_2 \dots Q'_k x_k \neg R(x_1, x_2, \dots, x_k, x)\}$
- $L_1 \in \Sigma_0^0 \Rightarrow L_1 = \{x|R(x)\} \Rightarrow \Sigma_0^0 = \mathbf{R}$
- $\bar{L_1} \in \Pi_0^0 \Rightarrow \bar{L_1} = \{x|\neg R(x)\} \Rightarrow \Pi_0^0 = \mathbf{R}$
- Prenex normal form
  - Tarski - Kuratowski algorithm
  - *praenexus* "tied or bound up in front", past participle of *praenectere*

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A proof for $\Sigma_1^0 = \mathbf{RE}$

$L \in \Sigma_1^0 \Rightarrow L = \{x | \exists y R(y,x)\}$

$\Sigma_1^0 \subseteq \mathbf{RE}$

- M_R TM which decides R
- We construct M_L as below
  - on input $\langle x \rangle$
  - run M_R for $\langle y = \varepsilon, x \rangle$ if it accepts M_L **accepts**, else
  - run M_R for the lexicographicaly next y

$\mathbf{RE} \subseteq \Sigma_1^0$

Theorem

Let $L \in \mathbf{RE}$, then

$$n \in L \iff (\exists (m_1, m_2, \ldots, m_k)(P(m_1, m_2, \ldots, m_k, n) = 0))$$

for some k and for P some Diophantine equation.

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for some k *and for* P *some Diophantine equation.*

A first step to show the hierarchy

Theorem

For all i, $\Sigma^0_{i+1} \supseteq \Sigma^0_i, \Pi^0_i$

Proof.

Use "dummy" quantifiers. □

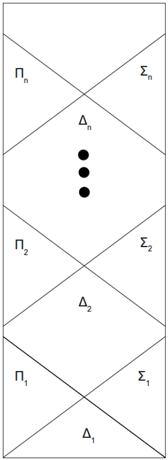

Figure: The Arithmetical Hierarchy

Upper bounds in AH and some known problems

- Showing upper bounds is generally an easy task
- $HALT \equiv H = \{\langle M, x\rangle \,|\, M(x)\downarrow\} = \{\langle M, x\rangle \,|\, \exists t\, M(x)\downarrow^{t}\}$
  - $HALT \in \Sigma_1^0 = \mathbf{RE}$
  - we already know that $HALT \notin \mathbf{R}$
- $K = \{\langle M\rangle \,|\, M(M)\downarrow\} = \{\langle M\rangle \,|\, \exists t\, M(M)\downarrow^{t}\} \in \Sigma_1^0$
  - $K = \{x | \varphi_x(x)\downarrow\} = \{x | \exists t\, \varphi_x(x)\downarrow^{t}\}$

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The power from below

Fact

We can enumerate languages in **RE** $= \Sigma^0_1$ *using a TM*

Question. Can we enumerate/decide languages of higher hierarchy?

Question. How much "stronger" we have to make a TM to be able to enumerate/decide a language in Σ^0_n?

Question. How can we make a TM "stronger"?

Answer. We will give to TMs the power to decide *difficult* problems (Oracles)

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Oracles and the AH

Let $L \in \Sigma_2^0$. Then $L = \{x | \exists y \forall z R(y,z,x)\}$ where R is a recursive predicate.

- We construct first the TM M,
  1. M gets an input $\langle y, x \rangle$
  2. then by dovetailing check for all z if $R(y,z,x) = 1$. If at any step $R = 0$ M **rejects**.
- We construct now the TM M_L,
  1. M_L gets an input $\langle x \rangle$
  2. lexicographicaly gets a y and put as input in M $\langle y, x \rangle$
  3. then M' goes into the special state, if the next state is $q_{"yes"}$ then we go to (2) and try the next y, if the next state is $q_{"no"}$ then then M' **accepts**.

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Oracles and the AH

Theorem

The languages in Σ_2^0 can be enumerated by a TM^H where H is an oracle for the Halting problem.

Can we do something similar to this with languages in Σ_n^0 for $n > 2$?

Yes, we can.

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Oracles and the AH

Theorem

The languages in Σ^0_2 can be enumerated by a TM^H where H is an oracle for the Halting problem.

Can we do something similar to this with languages in Σ^0_n for $n > 2$?

Yes, we can.

Let's Jump

Definition

Let A a language. Then we define $A' = K^A = \left\{x | \varphi_x^A(x)\downarrow\right\}$. A' is called *jump* of A, and $A^{(n)}$ is the nth jump of A.

- $\emptyset' := K = \{x | \varphi_x(x)\downarrow\}$.

Theorem

$\emptyset^{(n)}$ is $\leq_m$-complete for Σ_n^0, for $n \geq 1$.

Proof.

By induction. □

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We Jumping...

$\emptyset' = K$ is complete in $\Sigma_1^0 = \mathbf{RE}$. By induction it suffices to show that K^A is Σ_{n+1}^0-complete when A is Σ_n^0-complete.

$$K^A = \left\{x | \varphi_x^A(x)\right\} = \left\{x | (\exists t)\, \varphi_x^A(x) \downarrow^t\right\}$$

$K^A \in \Sigma_{n+1}^0$. Need to show that K^A is Σ_{n+1}^0-hard when A is Σ_n^0-hard.

- Let $B \in \Sigma_{n+1}^0$, $B = \{x | \exists y \langle x, y\rangle \in C\}$ where $C \in \Pi_n^0$.
- A is Σ_n^0-hard so $\bar{A}$ is Π_n^0-hard
- exists mapping $\sigma(\langle x, y\rangle) \in \bar{A} \iff \langle x, y\rangle \in C$

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Still Jumping...

- Now we define a mapping τ that is on x, $\tau(x)$ is the index of a TM M^A which on any input,
  - enumerates $y = 0, 1, 2, \ldots$
  - compute $\sigma(\langle x, y \rangle)$
  - ask oracle if $\sigma(\langle x, y \rangle) \notin A$ and if it says *yes* then M^A halt.

$$\begin{aligned} x \in B &\iff \exists x \, \langle x, y \rangle \in C \\ &\iff \exists y \, \sigma(\langle x, y \rangle) \notin A \\ &\iff \varphi^A_{\tau(x)}(\tau(x)) \downarrow \\ &\iff \tau(x) \in K^A \end{aligned}$$

So τ consists a reduction from B to K^A, and K^A is Σ^0_{n+1}-hard.

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A part of Jump's Theorem

Theorem

If A is $\leq_m$-complete for Σ^0_n, then $A' \notin \Sigma^0_n$.

Proof.

Suppose $A' = K^A \in \Sigma^0_n$. Because A is $\leq_m$-complete for Σ^0_n there is mapping σ s.t.

$$x \in K^A \iff \sigma(x) \in A$$

Let M^A be TM with an oracle for A that halts on y iff $\sigma(y) \in A$

$$\begin{aligned} \sigma\left(\left\langle M^A \right\rangle\right) \in A &\iff \left\langle M^A \right\rangle \in K^A \\ &\iff M^A\left(\left\langle M^A \right\rangle\right)\downarrow \\ &\iff \sigma\left(\left\langle M^A \right\rangle\right) \notin A \end{aligned}$$

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Theorem of Arithmetical Hierarchy

Theorem

Arithmetical Hierachy does not collapse.

Proof.

$\emptyset^{(n)} \in \Sigma^0_n \backslash \Pi^0_n$ and $\emptyset^{\overline{(n)}} \in \Pi^0_n \backslash \Sigma^0_n$. Then

$$(\forall n > 0)\left[\Delta^0_n \subset \Sigma^0_n \,\&\, \Delta^0_n \subset \Pi^0_n\right]$$

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Post's Theorem

Theorem

For $n \geq 0$,

1. $B \in \Sigma^0_{n+1} \iff B$ *is r.e. in some* Π^0_n *set* $\iff B$ *is r.e in some* Σ^0_n *set*
2. $\emptyset^{(n)}$ *is* Σ^0_n*-complete for* $n > 0$
3. $B \in \Sigma^0_{n+1} \iff$ *is r.e. in* $\emptyset^{(n)}$
4. $B \in \Delta^0_{n+1} \iff B \leq_T \emptyset^{(n)} \iff$ *is decided in* $\emptyset^{(n)}$

- A language in Σ^0_{n+1} can be enumerated by a TM^A where $A \in \Sigma^0_n$
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Lower bounds in AH

$$TOTAL = \{\langle M\rangle \mid L(M) = \Sigma^*\} = \{\langle M\rangle \mid \varphi_M \text{ is total}\}$$

- $TOTAL = \{\langle M\rangle \mid (\forall x \in \Sigma^*)(\exists t \in \mathbb{N})\, M(x)\downarrow^t\} \in \Pi_2^0$
- Let $A \in \Pi_2$
  - $x \in A \iff (\forall y)(\exists z)\, R(y,z,x)$
  - exists $f(x)$ s.t. $\varphi_{f(x)}(u) = \begin{cases} 0, & \text{if } (\forall y \le u)(\exists z)\, R(y,z,x) \\ \uparrow, & \text{otherwise} \end{cases}$
  - $x \in A \Rightarrow f(x)$ total
  - also if $x \in \bar{A} \Rightarrow L_{f(x)}$ finite (!) $\Rightarrow f(x)$ finite

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  - also if $x \in \bar{A} \Rightarrow L_{f(x)}$ finite (!) $\Rightarrow f(x)$ finite

Lower bounds in AH

$$TOTAL = \{\langle M \rangle \,|\, L(M) = \Sigma^*\} = \{\langle M \rangle \,|\, \varphi_M \text{ is total}\}$$

- $TOTAL = \{\langle M \rangle \,|\, (\forall x \in \Sigma^*)(\exists t \in \mathbb{N})\, M(x) \downarrow^t\} \in \Pi_2^0$
- Let $A \in \Pi_2$
  - $x \in A \iff (\forall y)(\exists z)\, R(y,z,x)$
  - exists $f(x)$ s.t. $\varphi_{f(x)}(u) = \begin{cases} 0, & \text{if } (\forall y \leq u)(\exists z)\, R(y,z,x) \\ \uparrow, & \text{otherwise} \end{cases}$
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$$TOTAL = \{\langle M\rangle \mid L(M) = \Sigma^*\} = \{\langle M\rangle \mid \varphi_M \text{ is total}\}$$

- $TOTAL = \{\langle M\rangle \mid (\forall x \in \Sigma^*)(\exists t \in \mathbb{N})\, M(x)\downarrow^t\} \in \Pi_2^0$
- Let $A \in \Pi_2$
  - $x \in A \iff (\forall y)(\exists z)\, R(y,z,x)$
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- $TOTAL = \{\langle M\rangle \mid (\forall x \in \Sigma^*)(\exists t \in \mathbb{N})\, M(x)\downarrow^t\} \in \Pi^0_2$
- Let $A \in \Pi_2$
  - $x \in A \iff (\forall y)(\exists z)\, R(y,z,x)$
  - exists $f(x)$ s.t. $\varphi_{f(x)}(u) = \begin{cases} 0, & \text{if } (\forall y \le u)(\exists z)\, R(y,z,x) \\ \uparrow, & \text{otherwise} \end{cases}$
  - $x \in A \Rightarrow f(x)$ total
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Lower bounds in AH

$INF = \{\langle M \rangle \,|\, L(M) \text{ is infinite } \}$

- $INF =$
  $\{\langle M \rangle \,|\, (\forall n \in \mathbb{N})\,(\exists t \in \mathbb{N}, x \in \Sigma^*)\,|\langle x \rangle| > n \Rightarrow M(x)\downarrow^t\} \in \Pi_2^0$
- A is Σ_n^0-complete iff $\bar{A}$ is Π_n^0-complete
- $\bar{INF} = FIN =$
  $\{\langle M \rangle \,|\, (\exists n \in \mathbb{N})\,(\forall t \in \mathbb{N}, x \in \Sigma^*)\,|\langle x \rangle| > n \Rightarrow M(x)\not\downarrow^t\} \in \Sigma_2^0.$
- In previous slide we showed that FIN is Σ_2^0-complete

Lower bounds in AH

$INF = \{\langle M \rangle \mid L(M) \text{ is infinite } \}$

- $INF =$
  $\{\langle M \rangle \mid (\forall n \in \mathbb{N})(\exists t \in \mathbb{N}, x \in \Sigma^*) |\langle x \rangle| > n \Rightarrow M(x) \downarrow^t\} \in \Pi_2^0$
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- In previous slide we showed that FIN is Σ_2^0-complete

More Problems

- The Riemann Hypothesis is in Π_1^0
- The Twin Prime Conjecture is in Π_2^0
- $\mathbf{P} \neq \mathbf{NP}$ is in Π_2^0

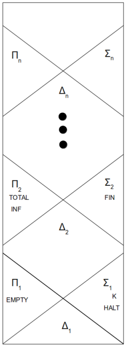

Figure: The Arithmetical Hierarchy

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