Function & Total Search Complexity Classes

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November 29, 2012

Function Problems

$L \in NP$

There is a polynomial-time decidable, polynomially balanced relation R_L such that for all strings x: there is a string y with $R_L(x, y)$ if and only if $x \in L$.

FL

Given x, find a string y such that $R_L(x, y)$ if such a string exists; if no such string exists, return "no".

Function Problems

Reductions

Functions Problems: $A \leq B$

- if x is an instance of A, then $R(x)$ is an instance of B.
- if there exists a solution for A with input x, then there exists a solution for B with input $R(x)$.
- if z is a solution for $R(x)$, then $S(z)$ is a solution for x.
- R, S are computable in logarithmic space.

$FP = FNP \iff P = NP$

- SAT can be solved in polynomial time if and only if FSAT can be solved in polynomial time
- FSAT is FNP-complete

Total Search Problems

Function Problem

1. Decision Problem: Decide if a solution exists ("yes", "no")
2. Search Problem: if "yes", find a solution

Total Search Problem

A "Total" FNP (TFNP) problem is an FNP problem where a solution is guaranteed to exist.

Total Search Problems

$$FP \subseteq TFNP \subseteq FNP$$

- $FP = TFNP \Rightarrow P = NP \cap coNP$
- $TFNP = FNP \Rightarrow NP = coNP$

Interesting to define classes of problems where solution is guaranteed to exist by a non-constructive proof.

Plan

1. Represent possible configurations with nodes.
2. Find a relation between nodes (Edges). The relation must be chosen so that the solutions are nodes with a special property (The non-consturctive proof helps!)

Total Search Problems

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PLS

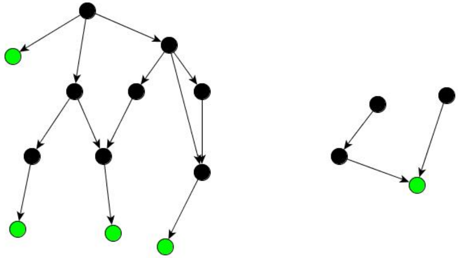


Figure: FIND SINK

Argument in Proof of Existence

Every finite directed acyclic graph has a sink.

PLS

Stable configuration for neural networks

Neural Network:

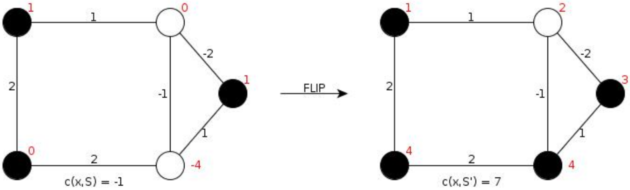
- $G = (V, E)$
- $S : V \to \{-1, 1\}$ (Nodes)
- Stable Configuration: $(\forall i \in V) S(i) \cdot \sum_{\{i,j\} \in E} S(j) w_{ij} \geq 0$

Define:

- Cost: $c(x, S) = \sum_{\{i,j\} \in E} S(i) S(j) w_{ij}$
- Neigborhood (Edges):

$$S' \in N(x, S) \iff (S' \in FLIP(S) \wedge (c(x, S') > c(x, S))$$

PLS



Stable Configuration for neural networks $\leq$ FIND SINK

Sinks $\subseteq$ *Solutions*: If node i is flipped and $S(i) \cdot \sum_{\{i,j\} \in E} S(j) w_{ij} = -\delta < 0$, then $c(x, S') = c(x, S) + 2\delta$.

PLS-complete Problems

- TSP, under the Kernighan-Lin neighborhood.
- MAX-CUT, under the flip neighborhood.
- Stable configuration for neural networks.

PPA

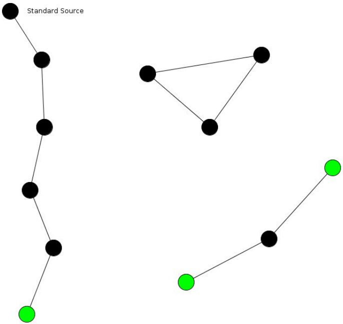
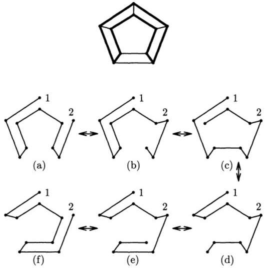


Figure: ODD DEGREE NODE

Argument in Proof of Existence

Any finite graph has an even number of odd-degree nodes *OR* All graphs of degree two or less have an even number of leaves.

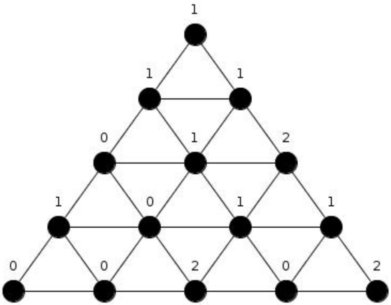


SMITH

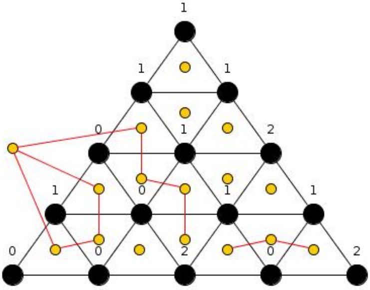
Given a graph G with odd degrees, and a Hamilton cycle, find another one.

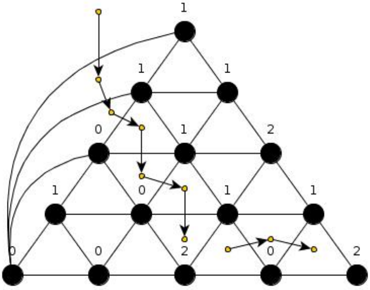
- Nodes: Hamilton Paths without edge $\{1, 2\}$.
- Edges: Add an edge to the endpoint ($\neq 1$) and break cycle in a unique way.

Sperner's Lemma



Sperner's Lemma





PPAD

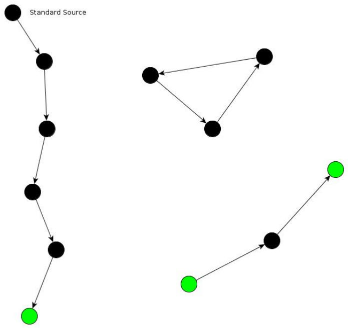


Figure: END OF THE LINE

Argument in Proof of Existence

If a finite directed graph has an unbalanced node (a vertex with different in-degree and out-degree), then it has another one.

PPAD-complete Problems

- 3D SPERNER
- BROUWER
- NASH

Hierarchy

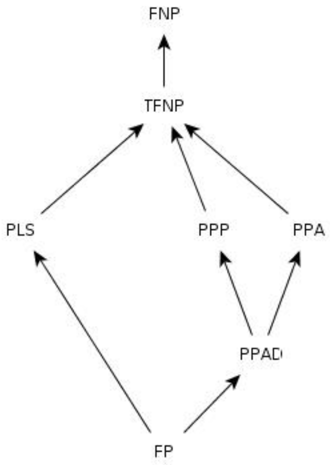


Figure: Search Classes

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