Dinur's Proof of the PCP Theorem

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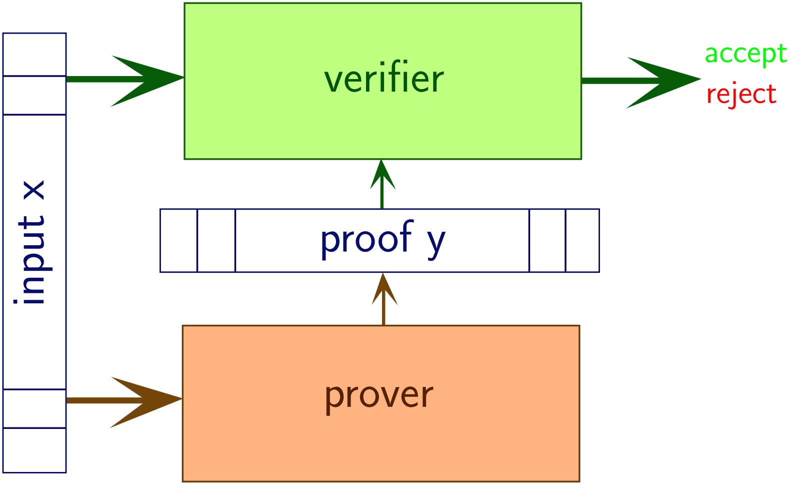
Characterization of NP



Characterization of NP

Prover

The prover of a language L has to output a correct proof y with $|y| \leq poly(|x|)$ when $x \in L$. When $x \notin L$ prover could output anything.

Verifier

The verifier of a language $L \in NP$ has to be efficient (polynomialy time bounded).

Characterization of NP

NP prover-verifier

For every language $L \in NP$ there exists a prover P and an efficient verifier V, that reads :

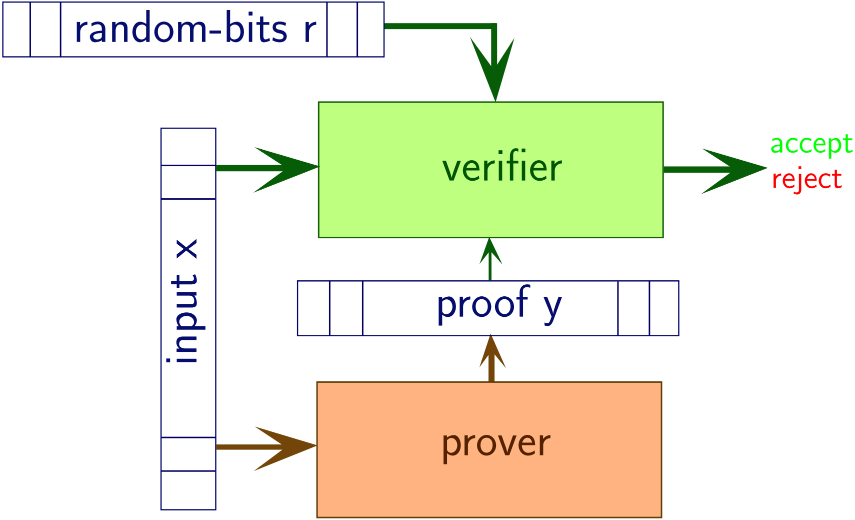
- the input string x
- the proof string y

and accepts or rejects such that :

Completeness : If $x \in L$ then P outputs a proof y such that V accepts.

Soundness : If $x \notin L$ then V rejects every proof y.

New characterization of NP [Arora, Safra]



New characterization of NP [Arora, Safra]

Probabilistically Checkable of Proofs

Definition : PCP $[p(n), q(n)]$

For every $L \in PCP[p(n),\ q(n)]$ there exists a prover P and an efficient verifier V, that reads :

- an input string x with length n
- **a random string r with length p(n)**
- **q(n) bits** randomly from the proof y

and accepts or rejects, such that :

Completeness : If $x \in L$ then P outputs a proof y such that V accepts **with probability 1**.

Soundness : If $x \notin L$ then V accepts **with probability** $\leq \frac{1}{2}$.

New characterization of NP [Arora, Safra]

Probabilistically Checkable of Proofs

Definition : PCP $[O(\log n), O(1)]$

For every $L \in PCP[O(\log n),\ O(1)]$ there exists a prover P and an efficient verifier V, that reads :

- an input string x with length n
- **a random string r with length c log n**
- **k bits** randomly from the proof y

with $c, k \in \mathbb{R}^+$ and accepts or rejects, such that :

Completeness : If $x \in L$ then P outputs a proof y such that V accepts **with probability 1**.

Soundness : If $x \notin L$ then V accepts **with probability** $\leq \frac{1}{2}$.

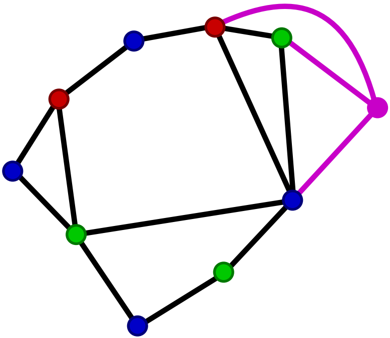
New characterization of NP [Arora, Safra]

Probabilistically Checkable of Proofs

PCP Theorem [Arora-Lund-Motwani-Sudan-Szegedy 92]

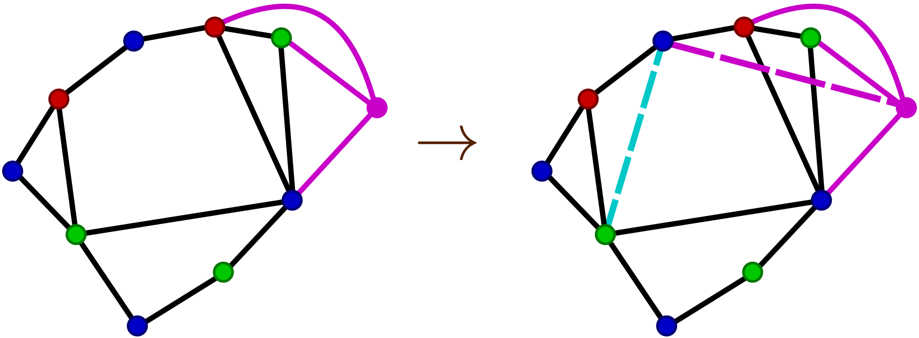
$$\mathbf{NP} = \mathbf{PCP}[\mathbf{O}(\mathbf{log\ n}), \mathbf{O}(\mathbf{1})]$$

Key Idea
Graph Coloring Problem - Constraint Graph Problem



Key Idea

Graph Coloring Problem - Constraint Graph Problem



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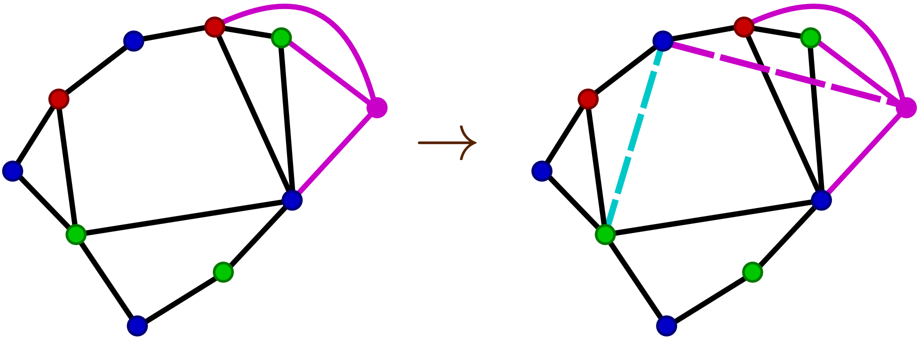
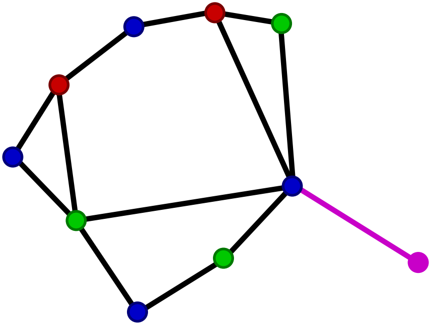


Key Idea

Add contraints for every path with length $\leq t$ in the graph G.

Good Idea but...



A Lemma on expanders...

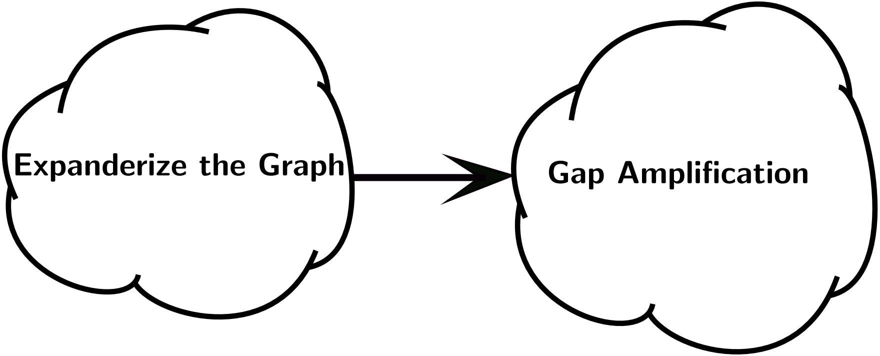
from Trevisan's lectures

Lemma

Let $G = (V, E)$ *be an expander and* $F \subseteq E$ *then for every path* p *in* G *with length* t

$$\Pr[p \text{ completely misses } F] \leq \left(1 - t\frac{|F|}{|E|}\right)$$

Proof Overview



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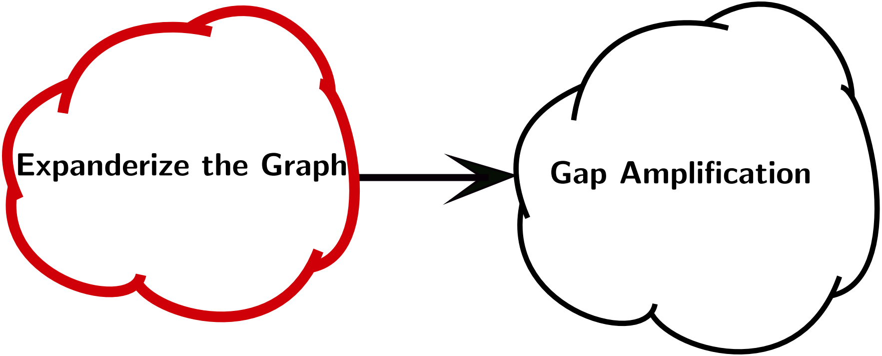

Introduction to Expanders

Definition

The edge expansion of a graph $G = (V, E)$, denoted by $\phi(G)$, is defined as

$$\phi(G) = \min_{S \subseteq V, |S| \leq |V|/2} \frac{|E(S, \bar{S})|}{|S|}$$

Introduction to Expanders

Lemma

There exists a constant ϕ_0 such that for every $n \in \mathbb{N}$ and $d < n$ there is an efficient algorithm to construct a $d-$regular graph G with $\phi(G) \geq \phi_0$.

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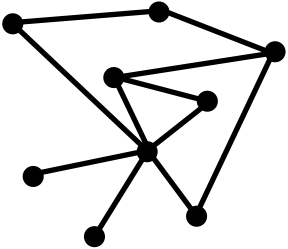
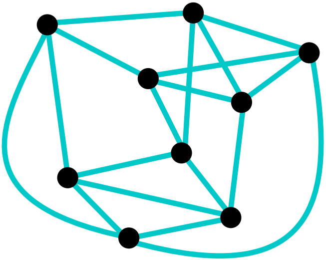
useful property: If a we add edges to a graph which is expander then the graph remains expander.

Union with expander



G

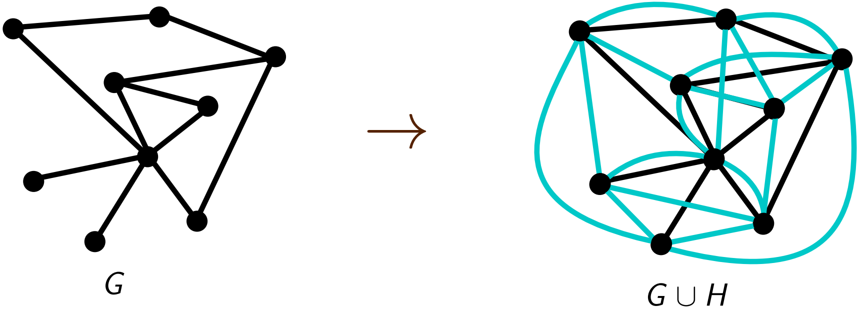
$\cup$



H

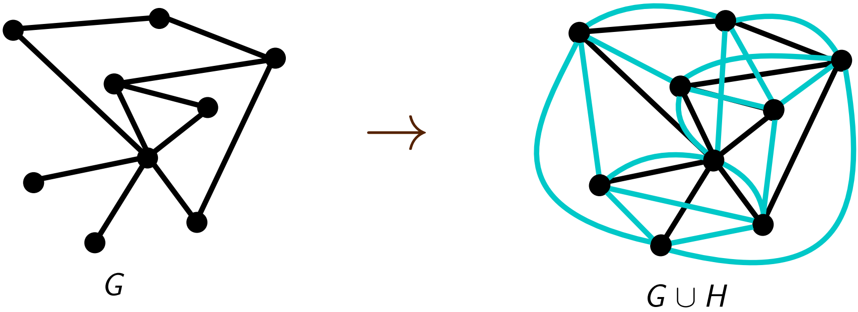
$d-$regular expander

Union with expander



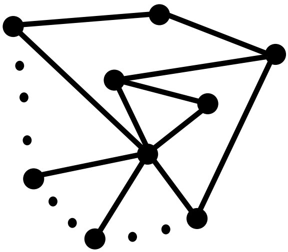
G is satifiable iff $G \cup H$ is satisfiable

Union with expander



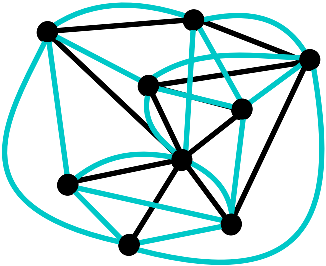
what about gap?

Union with expander



G
d−regular

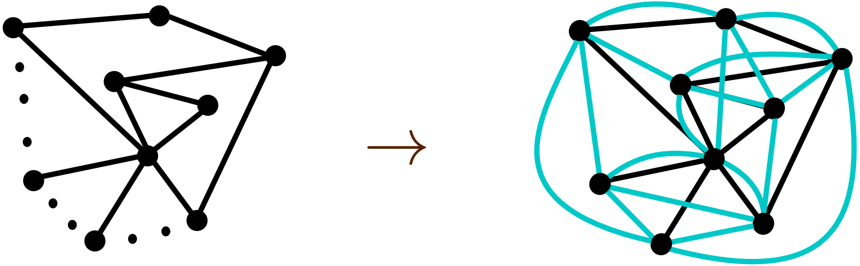
→



$G \cup H$
$2d$−regular

G is satifiable iff $G \cup H$ is satisfiable

Union with expander

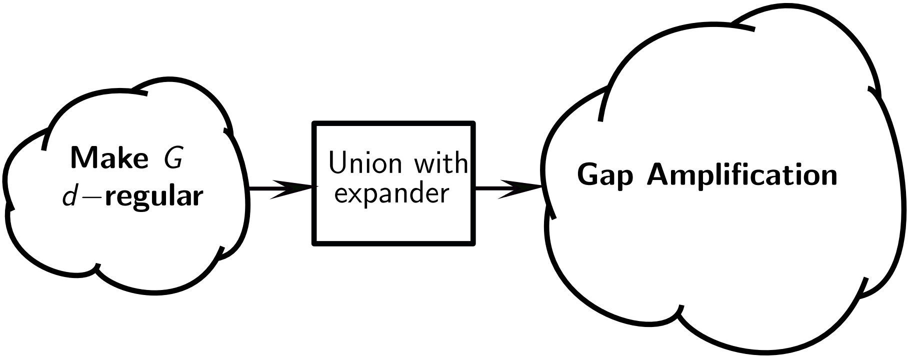


G
$d-$regular

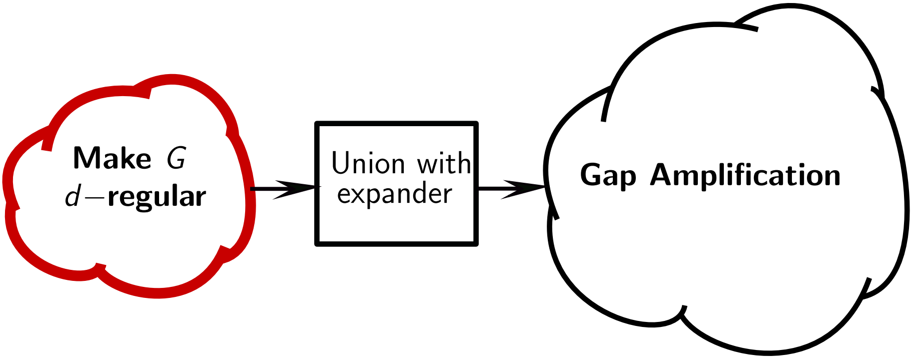
$G \cup H$
$2d-$regular

$$gap(G \cup H) \geq 1/2 gap(G)$$

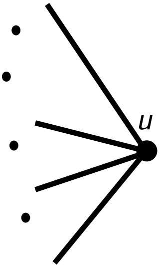
Proof Overview



Proof Overview



Make G $d-$regular
Papadimitriou and Yannakakis



Make G $d-$regular
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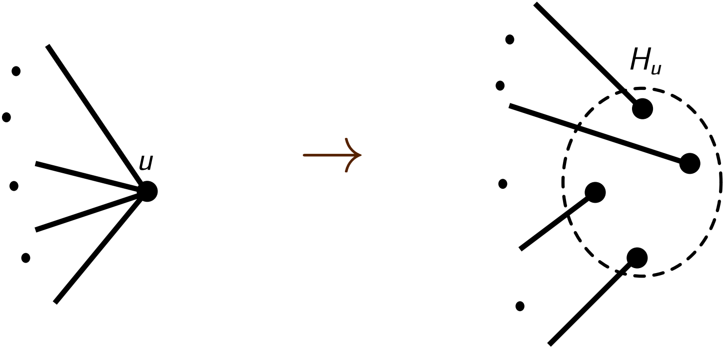

Make G $d-$regular

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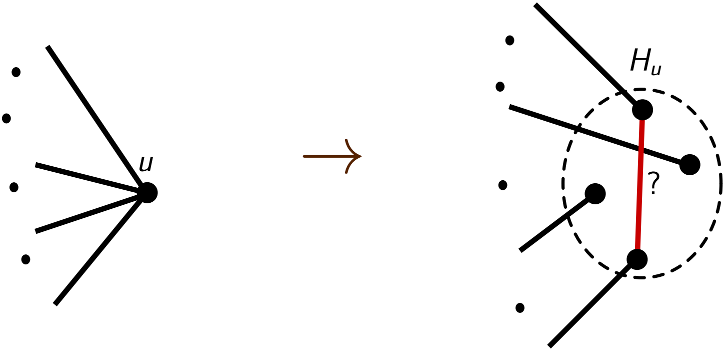
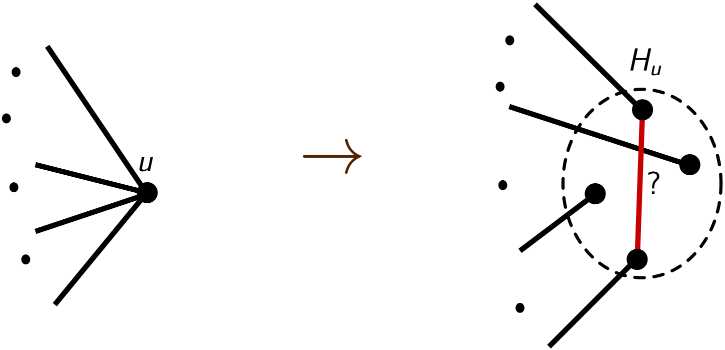


What kind of constraints in H_u ?

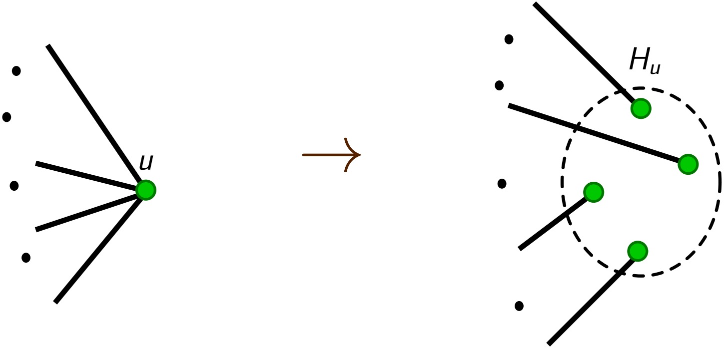
Make G $d-$regular

Papadimitriou and Yannakakis



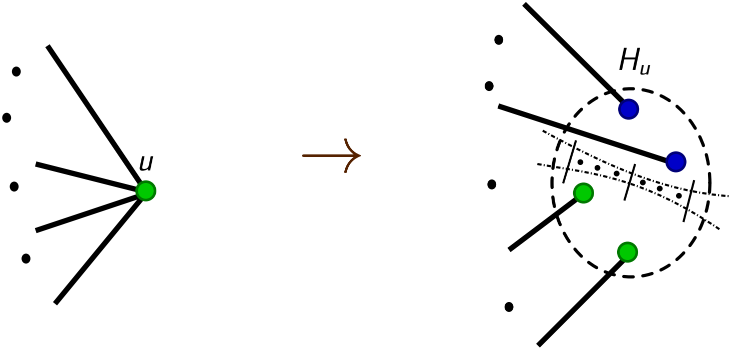
What kind of constraints in H_u ?
EQUALITY

Make G $d-$regular
Papadimitriou and Yannakakis



G is satifiable iff G' is satisfiable

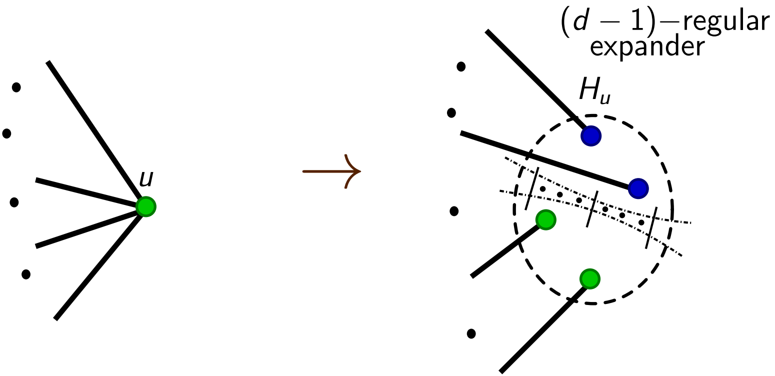
Make G $d-$regular
Papadimitriou and Yannakakis



What about gap?

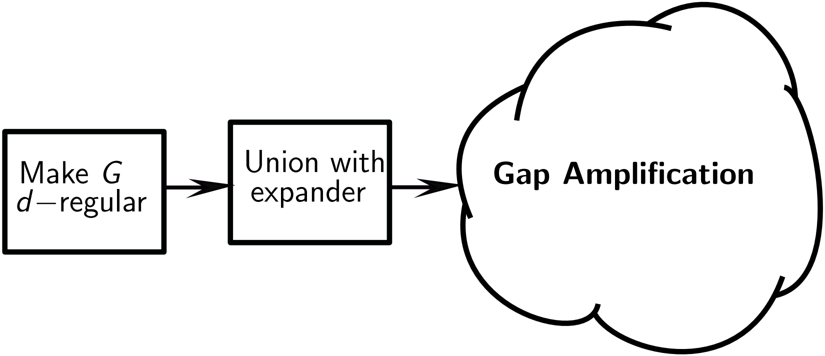
Make G $d-$regular

Papadimitriou and Yannakakis



$$gap(G') \geq 1/O(1)gap(G)$$

Proof Overview



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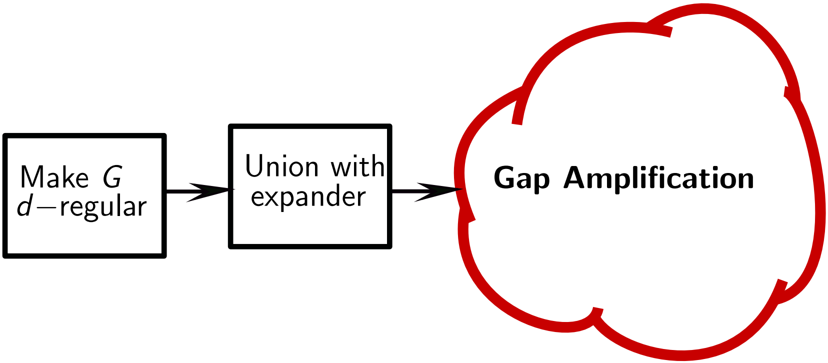


Gap Amplification

Key Idea

Add contraints for every path with length $\leq t$ in the graph G.

Gap Amplification

Key Idea

Add contraints for every path with length $\leq t$ in the graph G.

for t = 2

Our initial alphabet is $\Sigma = \{red, green, blue\}$.
Now every vertex u has an *opinion* about the color of the vertices in $N(u)$ and therefore the alphabet becomes $\Sigma' = \Sigma \times \Sigma^d$.
For every path $\{u, w, v\}$ with length 2 we add an edge $\{u, v\}$ with constraint : $c_{\{u,v\}}$ is true if the opinion u has about w is the same as the opinion v has about w.

Gap Amplification

Key Idea

Add contraints for every path with length $\leq t$ in the graph G.

for every t

Our initial alphabet is $\Sigma = \{red, green, blue\}$.
Now every vertex u has an *opinion* about the color of the vertices in $N(u)$ and therefore the alphabet becomes $\Sigma' = \Sigma \times \Sigma^d \times \cdots \times \Sigma^{d^t}$.
For every path $\{u, w, \ldots, v\}$ with length t we add an edge $\{u, v\}$ with constraint : $c_{\{u,v\}}$ is true if the opinion u has about every internal vertex w is the same as the opinion v has about w, and every internal edge of G is valid using this opinion.

Gap Amplification

Key Idea

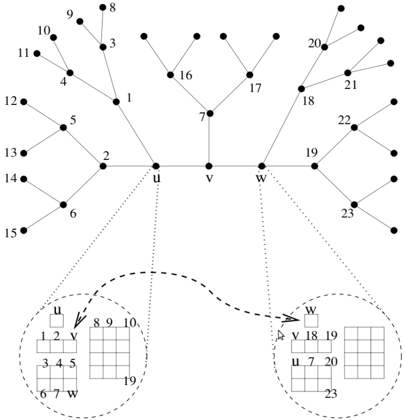
Add contraints for every path with length $\leq t$ in the graph G.

for every t

Our initial alphabet is $\Sigma = \{red, green, blue\}$.
Now every vertex u has an *opinion* about the color of the vertices in $N(u)$ and therefore the alphabet becomes $\Sigma' = \Sigma \times \Sigma^d \times \cdots \times \Sigma^{d^t}$.
For every path $\{u, w, \ldots, v\}$ with length t we add an edge $\{u, v\}$ with constraint : $c_{\{u,v\}}$ is true if the opinion u has about every internal vertex w is the same as the opinion v has about w, and every internal edge of G is valid using this opinion.

G is satifiable iff G' is satisfiable

Gap Amplification



Computational Complexity *Oded Goldreich*

Gap Amplification

We have now to use the following :

Lemma

Let $G = (V, E)$ be an expander and $F \subseteq E$ then for every path p in G with length t

$$\Pr[p \textit{ completely misses } F] \leq \left(1 - t\frac{|F|}{|E|}\right)$$

Where we set F the set of unsatisfied constraints in G.

Gap Amplification

Lemma

$$gap(G') \geq \frac{t}{O(1)} gap(G)$$

Proof Sketch

$$\begin{aligned} gap(G') &\geq 1/3 \Pr_{e'}[e' \text{ passes through } F] \\ &\geq 1/3(1 - \Pr_{e'}[e' \text{ completely misses } F]) \\ &\geq 1/3(1 - (t \cdot gap(G))) \\ &= \frac{t}{O(1)} gap(G) \end{aligned}$$

Gap Amplification

If we set $t = O(n)$ we have finished!

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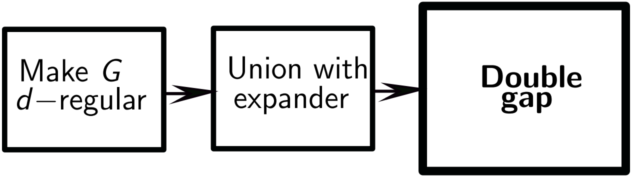
But we cannot do this because then $size(G') = O(d^t) = O(d^{O(n)})$ which is inefficient !!!

Gap Amplification

If we set $t = O(n)$ we have finished!
But we cannot do this because then $size(G') = O(d^t) = O(d^{O(n)})$ which is inefficient !!!
Therefore t must be a constant!

Proof Overview



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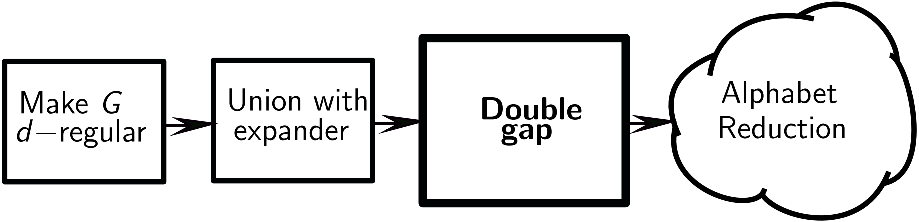

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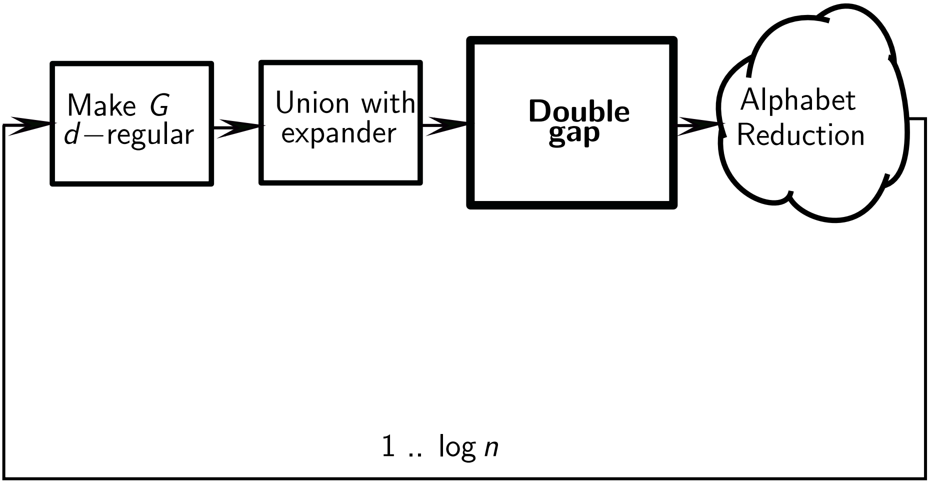

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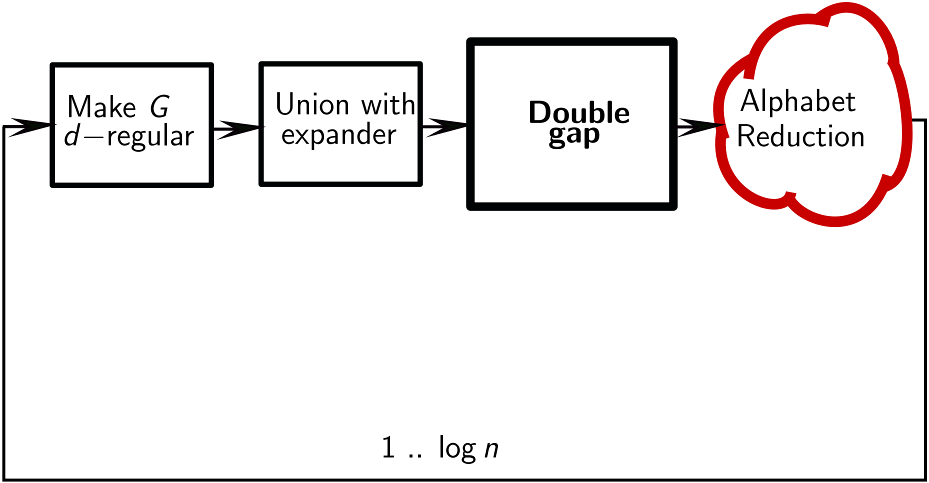

Alphabet Reduction

Effects of the Reduction

- Size increases a constant factor
- Gap decreases a constant factor
- Alphabet size redused to 16

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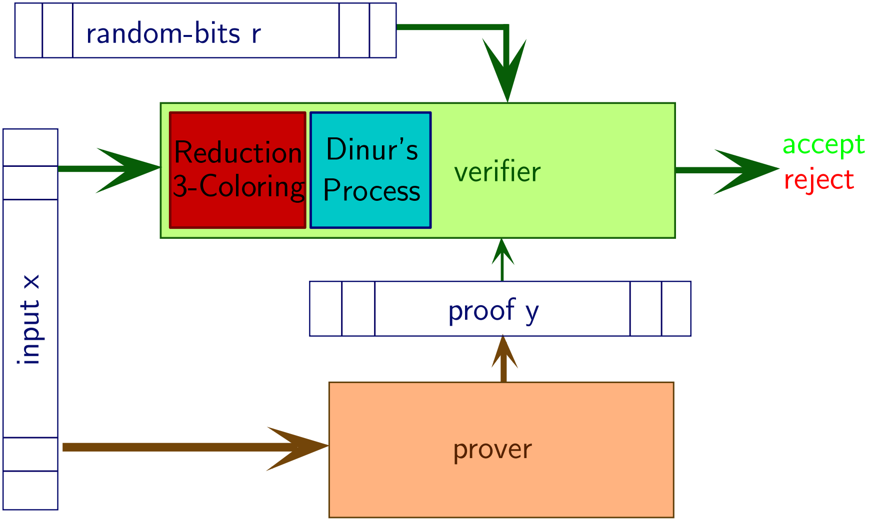
Proof Techniques

- Hadamard Codes
- Linearity Testing
- Fourier Analysis

Finishing the proof



Let's flip the coin

Coin

For every language L in NP there is a way to write proofs such that for every instance x :

- If $x \in L$ then there is a correct proof
- If $x \notin L$ then every proof has a lot of errors

Let's flip the coin

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For every language L in NP there is a way to write proofs such that for every instance x :

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One side

PCP Theorem

Let's flip the coin

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One side

PCP Theorem

The other side

Hardness of Approximation

Thanks!

Thank you! :)