Combinatory Complexity: Operators on Complexity Classes

Stathis Zachos   Aris Pagourtzis

Dept. of Computer Science, School of ECE, NTUA, 15780 Zografou, Greece
{zachos,pagour}@cs.ntua.gr

Abstract

Operators form a powerful tool that has been used to describe complexity classes, mainly those related to nondeterministic and probabilistic polynomial time Turing machines. Here we introduce new operators: the neutral operator $\mathcal{E}\cdot$, the complement operator $co\cdot$ and the symmetric operator $\Delta\cdot$. We show that many polynomial time complexity classes can be described by using these operators combined with other known operators. We investigate properties of operators when applied to complexity classes and to other operators. We thus develop a calculus of operators which offers new alternative, insightful descriptions of class inclusions and helps in proving new complexity theoretic results and in simplifying existing proofs of important theorems.

1 Introduction

We develop an algebraic manipulation calculus which facilitates proofs of inclusions between complexity classes. This calculus is based on operators that apply on complexity classes. Some of these have been introduced and used before [Zac82, HZ84, ZH83, Zac88, Tor88, AW93], while some other are completely new. Our basic operators are: $\mathcal{E}$, co, $\mathcal{N}$, $\mathcal{BP}$, $\mathcal{R}$, $\oplus$, $\mathcal{P}$, $\mathcal{U}$, Δ. Many others can be defined by combining these, e.g., $co\mathcal{N}$, $co\mathcal{R}$, $\Delta\mathcal{N}$, $\mathcal{ZP}$, $\mathcal{AM}$, $\mathcal{MA}$, $co\mathcal{AM}$, Σ_i^p, Π_i^p, $\Delta\Sigma_2^p$, etc.

We focus on complexity classes of languages related to polynomial time. It has been shown by several researchers that many of these classes can be defined by using suitable polynomially bounded quantifiers over predicates in the class **P**. Thus, for example, **NP** is the class of languages L for which there is an $L_1 \in \mathbf{P}$ such that

$$\begin{cases} x \in L \Rightarrow \exists y\, L_1(x,y) \\ x \notin L \Rightarrow \forall y\, \neg L_1(x,y) \end{cases}$$

This can be generalized by defining an operator $\mathcal{N}\cdot$ which, applied to a complexity class $\mathbf{C}$ gives a new complexity class $\mathcal{N}\cdot\mathbf{C}$. In [Zac82, Zac86] it was proven that probabilistic classes can also be defined in an analogous manner. Some of these ideas were used in [Tod89] where an important relation between the operators $\mathcal{BP}$ and $\oplus$ was shown.

We embark on a systematic study of these operators by introducing some new operators, namely the neutral operator $\mathcal{E}\cdot$, the complement operator $co\cdot$ and the intersection operator $\Delta\cdot$. These new operators play a key role in developing a calculus of operators on complexity classes that: (a) provides alternative insightful characterizations of known complexity classes,

(b) simplifies existing proofs of important theorems, and (c) leads to new inclusion results; some of them can be generalized for large families of classes.

Our paper is organized as follows: In section 2 we present the definitions of our operators and some simple properties of them. In section 3 we study inclusion properties of the operators and of the corresponding complexity classes. Finally, in section 4 we present a summary of new and old results under the operator description.

2 Basic operators and properties

Let $\mathbf{C}$ be any complexity class of languages involving polynomial time. We define the following operators, keeping in mind that all quantifiers are length bounded by a suitable polynomial $p(|x|)$, i.e., polynomial in size of the input x. For simplicity, we use $(\exists y)$ instead of $(\exists y \text{ with } |y| \leq p(|x|))$ and $(\forall y)$ similarly. We also assume $\Sigma = \{0, 1\}$ and $x, y, \ldots \in \Sigma^*$.

We denote languages by L, sometimes with a subscript. Our operators apply on classes of languages, giving classes of languages, or on other operators, giving operators.

Definition 1. The identity operator $\mathcal{E}$ (neutral element):

(i) $L \in \mathcal{E} \cdot \mathbf{C}$: there is an $L_1 \in \mathbf{C}$ such that $\begin{cases} x \in L \Rightarrow \forall y \, L_1(x, y) \\ x \notin L \Rightarrow \forall y \, \neg L_1(x, y) \end{cases}$

(ii) $(\mathcal{E} \cdot \mathcal{O}) \cdot \mathbf{C} := \mathcal{O} \cdot \mathbf{C}$ for any operator $\mathcal{O}$.

Fact. $\mathcal{E} \cdot \mathbf{C} = \mathbf{C}$.

Definition 2. The complement operator co:

(i) $L \in co \cdot \mathbf{C}$: there is an $L_1 \in \mathbf{C}$ such that $\begin{cases} x \in L \Rightarrow \forall y \, \neg L_1(x, y) \\ x \notin L \Rightarrow \forall y \, L_1(x, y) \end{cases}$

(ii) $(co \cdot \mathcal{O}) \cdot \mathbf{C} := co \cdot (\mathcal{O} \cdot (co \cdot \mathbf{C}))$ for any operator $\mathcal{O}$.

Some properties of co:
$co \cdot \mathbf{C} = \mathbf{coC}$, $\quad co \cdot (co \cdot \mathbf{C}) = \mathbf{C}$,
$(co \cdot (\mathcal{O} \cdot \mathbf{C})) := (co \cdot \mathcal{O}) \cdot (co \cdot \mathbf{C})$ for any operator $\mathcal{O}$,
$(co \cdot \mathcal{E}) \cdot \mathbf{C} = co \cdot (\mathcal{E} \cdot (co \cdot \mathbf{C})) = co \cdot (co \cdot \mathbf{C}) = \mathbf{C} = \mathcal{E} \cdot \mathbf{C}$,
$(\mathcal{E} \cdot co) \cdot \mathbf{C} = co \cdot \mathbf{C}$, $\quad (co \cdot co) \cdot \mathbf{C} = co \cdot (co \cdot (co \cdot \mathbf{C})) = co \cdot \mathbf{C}$.

Examples. $co \cdot \mathbf{NP} = \mathbf{coNP}$, $co \cdot \mathbf{P} = \mathbf{P}$.

Definition 3. The nondeterministic operator $\mathcal{N}$:

(i) $L \in \mathcal{N} \cdot \mathbf{C}$: there is an $L_1 \in \mathbf{C}$ such that $\begin{cases} x \in L \Rightarrow \exists y \, L_1(x, y) \\ x \notin L \Rightarrow \forall y \, \neg L_1(x, y) \end{cases}$

(ii) $(\mathcal{N} \cdot \mathcal{O}) \cdot \mathbf{C} := \mathcal{N} \cdot (\mathcal{O} \cdot \mathbf{C})$ for any operator $\mathcal{O}$.

Some properties of $\mathcal{N}$*:*
$\mathcal{N}\cdot\mathcal{N}\cdot\mathbf{C}=\mathcal{N}\cdot\mathbf{C}$, $\quad(\mathcal{E}\cdot\mathcal{N})\cdot\mathbf{C}=\mathcal{N}\cdot\mathbf{C}$, $\quad(\mathcal{N}\cdot\mathcal{E})\cdot\mathbf{C}=\mathcal{N}\cdot\mathbf{C}$,
$(co\cdot\mathcal{N})\cdot\mathbf{C}=co\cdot(\mathcal{N}\cdot(co\cdot\mathbf{C}))$, $\quad(\mathcal{N}\cdot co)\cdot\mathbf{C}=\mathcal{N}\cdot(co\cdot\mathbf{C})=co\cdot((co\cdot\mathcal{N})\cdot\mathbf{C})$

Examples. $\mathcal{N}\cdot\mathbf{P}=\mathbf{NP}$, $(co\cdot\mathcal{N})\cdot\mathbf{P}=\mathbf{coNP}=co\cdot(\mathcal{N}\cdot\mathbf{P})$, i.e., here parentheses are not needed.
Similarly $\mathcal{N}\cdot co\cdot\mathcal{N}\cdot\mathbf{P}=\Sigma_2^p$, $(co\cdot\mathcal{N})\cdot\mathcal{N}\cdot\mathbf{P}=\Pi_2^p$.

We use the "$\exists^+$" overwhelming majority quantifier introduced in [Zac82, ZH83]. The meaning of the complexity class is robust under perturbation of the majority threshold, i.e., it is immaterial whether the threshold is bounded away from 1/2 by ε or even by $1/p(|x|)$ or whether it is 3/4 or 99% or even $1-(1/2^{p(|x|)})$.

Definition 4. The bounded probabilistic operator $\mathcal{BP}$:

(i) $L\in\mathcal{BP}\cdot\mathbf{C}$: there is an $L_1\in\mathbf{C}$ such that $\begin{cases}x\in L\Rightarrow\exists^+y\,L_1(x,y)\\ x\notin L\Rightarrow\exists^+y\,\neg L_1(x,y)\end{cases}$

(ii) $(\mathcal{BP}\cdot\mathcal{O})\cdot\mathbf{C}:=\mathcal{BP}\cdot(\mathcal{O}\cdot\mathbf{C})$ for any operator $\mathcal{O}$.

Some properties of $\mathcal{BP}$*:*
$\mathcal{BP}\cdot\mathbf{C}=(co\cdot\mathcal{BP})\cdot\mathbf{C}=co\cdot(\mathcal{BP}\cdot(co\cdot\mathbf{C}))$,
$\mathcal{BP}\cdot(co\cdot\mathbf{C})=(co\cdot\mathcal{BP})\cdot(co\cdot\mathbf{C})=co\cdot(\mathcal{BP}\cdot\mathbf{C})$.

Examples. $\mathcal{BP}\cdot\mathbf{P}=\mathbf{BPP}$, $\mathcal{BP}\cdot\mathcal{N}\cdot\mathbf{P}=\mathbf{AM}$, $\mathcal{N}\cdot\mathcal{BP}\cdot\mathbf{P}=\mathbf{MA}$,
where **MA** and **AM** are the well known Arthur-Merlin and Merlin-Arthur classes [Zac88, BM88].

Definition 5. The random operator $\mathcal{R}$:

(i) $L\in\mathcal{R}\cdot\mathbf{C}$: there is an $L_1\in\mathbf{C}$ such that $\begin{cases}x\in L\Rightarrow\exists^+y\,L_1(x,y)\\ x\notin L\Rightarrow\forall y\,\neg L_1(x,y)\end{cases}$

(ii) $(\mathcal{R}\cdot\mathcal{O})\cdot\mathbf{C}:=\mathcal{R}\cdot(\mathcal{O}\cdot\mathbf{C})$.

Some properties of $\mathcal{R}$*:*
$(co\cdot\mathcal{R})\cdot\mathbf{C}=co\cdot(\mathcal{R}\cdot(co\cdot\mathbf{C}))$, $\quad co\cdot(\mathcal{R}\cdot\mathbf{C})=(co\cdot\mathcal{R})\cdot(co\cdot C)$.

Proposition 1.
$\mathcal{N}\cdot co\cdot\mathcal{R}\cdot\mathbf{C}=\mathcal{N}\cdot\mathcal{BP}\cdot\mathbf{C}$, $\quad(co\cdot\mathcal{R})\cdot\mathcal{N}\cdot\mathbf{C}=\mathcal{BP}\cdot\mathcal{N}\cdot\mathbf{C}$,
$\mathbf{coMA}=co\cdot(\mathcal{N}\cdot co\cdot\mathcal{R}\cdot\mathbf{P})=(co\cdot\mathcal{N})\cdot\mathcal{R}\cdot\mathbf{P}$.

Examples. $\mathcal{R}\cdot\mathbf{P}=\mathbf{RP}$, $co\cdot\mathcal{R}\cdot\mathbf{P}=\mathbf{coRP}$

Definition 6. The parity operator $\oplus\cdot$:

(i) $L\in\oplus\cdot\mathbf{C}$: there is an $L_1\in\mathbf{C}$ such that $\begin{cases}x\in L\Rightarrow\oplus y\,L_1(x,y)\\ x\notin L\Rightarrow\neg\oplus y\,L_1(x,y)\end{cases}$

(ii) $(\oplus\cdot\mathcal{O})\cdot\mathbf{C}:=\oplus\cdot(\mathcal{O}\cdot\mathbf{C})$

Some properties of $\oplus\cdot$*:*
$\oplus\cdot\mathbf{C}=\oplus(co\cdot\mathbf{C})$, $\quad(co\cdot\oplus)\cdot\mathbf{C}=co\cdot(\oplus\cdot(co\cdot\mathbf{C}))=co\cdot(\oplus\cdot\mathbf{C})$.

Proposition 2. $\oplus\cdot\mathbf{C}=co\cdot(\oplus\cdot\mathbf{C})$

Definition 7. The unambiguous (unique) operator $\mathcal{U}$:

(i) $L \in \mathcal{U} \cdot \mathbf{C}$: there is an $L_1 \in \mathbf{C}$ such that $\begin{cases} x \in L \Rightarrow \exists!\, y\, L_1(x,y) \\ x \notin L \Rightarrow \forall y\, \neg L_1(x,y) \end{cases}$

(ii) $(\mathcal{U} \cdot \mathcal{O}) \cdot \mathbf{C} := \mathcal{U} \cdot (\mathcal{O} \cdot \mathbf{C})$

Some properties of $\mathcal{U}$*:*
$\mathcal{U} \cdot \mathbf{C} \subseteq \oplus \cdot \mathbf{C}$, $\mathcal{U} \cdot \mathbf{C} \subseteq \mathcal{N} \cdot \mathbf{C}$, $(co \cdot \mathcal{U}) \cdot \mathbf{C} = co \cdot (\mathcal{U} \cdot (co \cdot \mathbf{C}))$, $(co \cdot \mathcal{U}) \cdot (co \cdot \mathbf{C}) = co \cdot (\mathcal{U} \cdot \mathbf{C})$.

We use the "$\exists_{>1/2}$" majority quantifier introduced in [Zac82] (denoted as a reverse "R" there). In contrast to the overwhelming majority operator "$\exists^{+}$", the majority operator does not seem to be robust under perturbation of the majority threshold, i.e. it is not likely that $\mathcal{P} \cdot \mathbf{C} \subseteq \mathcal{BP} \cdot \mathbf{C}$.

Definition 8. The probabilistic operator $\mathcal{P}$:

(i) $L \in \mathcal{P} \cdot \mathbf{C}$: there is an $L_1 \in \mathbf{C}$ such that $\begin{cases} x \in L \Rightarrow \exists_{>1/2}\, y\, L_1(x,y) \\ x \notin L \Rightarrow \exists_{>1/2}\, y\, \neg L_1(x,y) \end{cases}$

(ii) $(\mathcal{P} \cdot \mathcal{O}) \cdot \mathbf{C} := \mathcal{P} \cdot (\mathcal{O} \cdot \mathbf{C})$

Some properties of $\mathcal{P}$*:*
$co \cdot (\mathcal{P} \cdot \mathbf{C}) = (co \cdot \mathcal{P}) \cdot (co \cdot \mathbf{C}) = \mathcal{P} \cdot (co \cdot \mathbf{C})$, $(co \cdot \mathcal{P}) \cdot \mathbf{C} = co \cdot (\mathcal{P} \cdot (co \cdot \mathbf{C}))$.

Examples. $\mathcal{P} \cdot \mathbf{P} = \mathbf{PP}$, $co \cdot \mathcal{P} \cdot \mathbf{P} = \mathbf{PP}$

Definable operators:
$\Sigma_2^p \cdot \mathbf{C} = \mathcal{N} \cdot (co \cdot \mathcal{N}) \cdot \mathbf{C}$,
$\Pi_2^p \cdot \mathbf{C} = (co \cdot \mathcal{N}) \cdot \mathcal{N} \cdot \mathbf{C}$,
$\mathcal{MA} \cdot \mathbf{C} = \mathcal{N} \cdot (co \cdot \mathcal{R}) \cdot \mathbf{C} = \mathcal{N} \cdot \mathcal{BP} \cdot \mathbf{C}$,
$\mathcal{AM} \cdot \mathbf{C} = (co \cdot \mathcal{R}) \cdot \mathcal{N} \cdot \mathbf{C} = \mathcal{BP} \cdot \mathcal{N} \cdot \mathbf{C}$.

Definition 9. The intersection operator Δ:

(i) $\Delta \cdot \mathbf{C} := \mathbf{C} \cap co \cdot \mathbf{C}$

(ii) $(\Delta \cdot \mathcal{O}) \cdot \mathbf{C} := \mathcal{O} \cdot (co \cdot \mathbf{C}) \cap (co \cdot \mathcal{O}) \cdot \mathbf{C}$

Some properties of Δ*:*
$\Delta \cdot \mathbf{C} = \Delta \cdot (co \cdot \mathbf{C})$, $(\Delta \cdot \mathcal{O}) \cdot \mathbf{C} := \Delta \cdot (\mathcal{O} \cdot (co \cdot \mathbf{C}))$.

Examples. $\Delta \cdot \mathbf{P} = \mathbf{P}$, $\Delta \cdot \mathbf{NP} = \mathbf{NP} \cap \mathbf{coNP}$, $\Delta \cdot \mathbf{RP} = \mathbf{RP} \cap \mathbf{coRP} = \mathbf{ZPP}$.

New definable operators:
$\mathcal{ZP} \cdot \mathbf{C} = (\Delta \cdot \mathcal{R}) \cdot \mathbf{C}$.